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LONG CYCLES IN RESIDENTIAL BUILDING: AN EXPLANATION[1]

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INTRODUCTION

THIS PAPER IS DEVOTED TO an investigation of the long cycles in residential building. It is now generally known from the statistical data that gradually became available, that residential building in the United States for more than a hundred years has shown cyclical fluctuations of a duration varying from 15 to 22 years.[2] This "building cycle" is an example of a cycle independent of the variations in general business activity. Such cycles have been observed for a number of commodities, Hanau's famous pig cycle being one of the first cases to have drawn the attention of economists. These cycles are of primary importance for the study of the business cycle because they represent the simplest cases of endogenous cycles. The research into these cycles has without doubt greatly stimulated attempts to find an endogenous explanation of the business cycle through the creation of "models," one of the latest contributions in this field being Tinbergen's "closed system of equations" as developed in his second League of Nations publication.[3]

The explanation of the cycles as observed for hogs, potatoes, beef cattle, and shipbuilding is not the same in all cases. The common characteristic is that the cyclical fluctuations are governed by internal causes. Exogenous influences may initiate the fluctuations. The length of the waves, however, and their damping or antidamping, are due not to external causes but to the endogenous mechanism of quantities and prices. For the purposes of this article it is sufficient to distinguish roughly between two groups of commodities: (1) perishable consumption goods, and (2) durable consumption goods and capital goods. In

[1] Read before the Econometric Society at Philadelphia, Pennsylvania, December, 28, 1939. The paper was prepared during a stay at the National Bureau of Economic Research, under a Rockefeller Fellowship. The author feels indebted to the National Bureau for facilities for work and towards the members of its staff for valuable comments.

[2] Cf. Warren and Pearson, *World Prices and the Building Industry*, New York, 1937, p. 109.

[3] J. Tinbergen, *Business Cycles in the United States, 1919–1932*, Geneva, 1939.

the first case the explanation of the cycle is given by the so-called cobweb theorem. According to this theory the cyclical fluctuations observed for a number of these commodities are explained by the lag by which production follows fluctuations in price. It can easily be shown that the period of the cycles is equal to twice that lag and is not affected by the elasticities of demand and supply. For a discussion of the various cases which may arise and the conditions that have to be fulfilled in order that the cobweb theorem may be applicable, we may refer to the literature on this subject.[4]

In the case of durable consumption goods or of capital goods the explanation has to be different. Here it is not production in a certain time unit, but the total available supply, which affects the price of the services rendered by these goods. In turn production follows the fluctuations of price with a certain time lag which represents the time required for the entrepreneurs to make their decisions plus the period necessary for production. The mechanism may be described by the following equation:

$$f'(t) = -af(t-\theta), \tag{1}$$

in which $f(t)$ represents the total available supply at the time t and the derivative $f'(t)$ represents its net increase. $f(t)$ is seen as a deviation from a "normal level." It is assumed that a and θ are constant for the period considered. The lag θ equals the sum of the lags involved, that is, it is equal to the lag between the fluctuations of $f(t)$ and the price level plus the lag by which the price precedes the production $f'(t)$. A negative value of $f(t)$ will be followed, after a period θ, by a positive value of $f'(t)$. A positive value of $f(t)$, after a time period θ, will be followed by a negative value of $f'(t)$.

The purpose of this study is to investigate whether equation (1), used for the first time by Tinbergen in his analysis of the shipbuilding cycle,[5] can also be used to explain the cycles in residential building.

The solutions of (1) can easily be given and the conditions developed under which they will be periodic functions. The character of the solution will depend on the product of a and θ. Quoting from Tinbergen, who also gives the mathematical derivations, the following cases may be distinguished. Putting $a\theta = B$, the solutions will be:

$$B < \frac{1}{e} \cdots, \text{ nonperiodic (exponential function),}$$

[4] E.g., M. Ezekiel, "The Cobweb Theorem," *Quarterly Journal of Economics*, February, 1938, pp. 255–80.

[5] J. Tinbergen, "Ein Schiffbauzyklus?" *Weltwirtschaftliches Archiv*, July, 1931, p. 152.

$$\frac{1}{e} < B < \frac{\pi}{2} \cdots, \text{ damped oscillations, period} > 4\theta,$$

$$B = \frac{\pi}{2} \cdots, \text{ pure sine function, period} = 4\theta,$$

$$B > \frac{\pi}{2} \cdots, \text{ explosive oscillations, period} < 4\theta.$$

If the initial values over the first time interval also are given, the solution is completely determined. These initial values will mainly determine the amplitude of the solutions, whereas the wave length and the damping degree depend on a and θ. We occasionally expect fluctuations of a very large amplitude, e.g., after a war, when a large destruction of capital goods has taken place, as occurred in the case of shipbuilding during the last war, or when the building of new units has been badly impaired through lack of raw materials or government interference, as has been the case in the building industry in the United States in 1917–1918.[6]

SCOPE OF THE ANALYSIS

So far only general principles have been discussed. Several questions arise when the previous scheme is applied to a special case, e.g., to the long cycles in residential building. It goes without saying that certain general conditions must be fulfilled, e.g., that pure competition shall prevail. Among the problems which require careful consideration are the following: In equation (1) the function $f(t)$, as mentioned before, indicates deviations from a certain normal level. What then must be understood by a "normal level" and how can it be determined statistically?

In the analysis of the shipbuilding cycle Tinbergen assumed that the total tonnage which can be considered "normal" is given by a trend for which a second-degree parabola was chosen. In the case of residential building, however, this procedure would be too inexact as the average duration of the cycle is here much longer and the available statistical material covers a much shorter period. It will, therefore, be assumed that for each year the normal need for shelter is given by the total number of nonfarm families plus a constant "normal" vacancy percentage.

It is advisable not to limit ourselves to equation (1) but to consider each of the steps leading to that equation. This may show more

[6] Schumpeter's remark (*Business Cycles*, Vol. II, p. 534), that these cycles cannot be called endogenous as they depend on some disturbance that starts them, implies a notion of "endogenous" cycles that is certainly too limited to be of much use in business-cycle research.

clearly the causal relationships involved and may help in detecting other variables in the mechanism. Besides it would be incorrect to try to estimate the constants a and θ by computing a simple regression equation involving the variables $f'(t)$ and $f(t)$ only, because a correlation analysis is not valid if one does not take into account all the variables which determine the phenomenon to be explained.

It is implied above that a total available supply below normal would lead to a rise in the level of rents; this higher level of rents would in turn result in an increase in new building. Now it may be remarked that the prospective builder will take into account not only the rent level but also elements of costs: building costs, interest rates, taxes, etc. Moreover new building may be influenced by such factors as are summarized in the so-called acceleration principle. It is furthermore reasonable to expect that rents themselves will be determined not only by the total available supply of houses in relation to the number of families, but also by such factors as average family income and the remaining items in the cost of living. Only rents, and perhaps also the foreclosure rate which will be examined later, represent endogenous variables. All other variables introduced must be considered as exogenous because they are mainly influenced by factors outside the building industry and the housing market.

The present study is limited to residential building. Commercial building does not show a pronounced endogenous cycle and seems to be determined to a large extent by the general business situation. In the same way construction of farms will depend mainly upon the situation in agriculture. Public construction is determined by the government, whose decisions may depend on the budget situation or in times of depression on its attitude towards a policy of public works.

There are reasons to believe that it may be incorrect to consider total nonfarm residential construction as a homogeneous economic variable. The housing market shows important imperfections. It is necessarily limited geographically and indeed should be considered as composed of a large number of markets with very restricted boundaries. In each of these markets factors may prevail which do not occur in others. We need not be surprised, therefore, to find that the fluctuations in residential construction show considerable regional differences. Whereas for the country as a whole residential building reached its peak in 1925, construction was at its height in the Pacific region in 1923 and in the Middle Atlantic states not until 1927.[7]

Another point that may deserve attention is the fact that for the country as a whole the various types of dwellings do not show similar

[7] Cf. D. L. Wickens and R. R. Foster, "Nonfarm Residential Construction, 1920–1936," National Bureau of Economic Research, *Bulletin* 63, p. 6–9.

fluctuations. The one- and two-family dwellings show a closely similar movement, whereas the construction of apartments shows a distinctly different development, its peak not being reached before 1928. The factors which govern the construction of apartments must, therefore, differ from the factors by which building of one- and two-family dwellings is influenced.

Despite these variations the present analysis will attempt to deal with residential building as a whole. As we shall assume that the equations involved are linear, we may consider our final results as the arithmetic sum of a number of equations for all geographic units. If we may assume that in all parts of the country the elasticities with respect to the variables determining residential construction are equal, the sum total of the regional equations will be identically the same as the equation found for the country as a whole, because all equations will have their due weights. This hypothesis implies that the regional differences in residential building are conceived of as caused only by differences between the exogenous factors: differences in the fluctuations of average income, of building costs, of population growth, etc. The justification of the grouping together of the one- and two-family dwellings and the apartments is based on the assumption that the relationships are materially the same, but differ with respect to the intensities of the reactions and with respect to the time lags. Our final results, therefore, will represent average elasticities and average time lags. At the end of our analysis we shall have to reconsider these assumptions.

LITERATURE

In recent years a number of interesting studies have been published on the building industry, among which some are limited to residential construction. So far, however, relatively little work has been done on the explanation of the long cycles. The available studies may roughly be classified into three groups.

1. Studies containing results of statistical research, such as those of Riggleman,[8] Warren and Pearson,[9] Wickens and Foster,[10] and the Department of Commerce.[11] Professional work in this field should also be included, e.g., the figures published by the F. W. Dodge Corporation and by the Real Estate Analysts, Inc.

2. Studies of a mainly descriptive character, such as those of New-

[8] J. R. Riggleman, "Building Cycles in the United States," *Journal of the American Statistical Association*, Vol. 28, 1933, p. 174.

[9] Cf. note 2.

[10] Cf. note 7.

[11] *Construction Activity in the United States, 1915–1937*, U. S. Department of Commerce, Bureau of Foreign and Domestic Commerce, Washington, 1938.

man[12] and Long.[13] These studies are especially useful as they provide the necessary facts and contain suggestions for an econometric approach to the problem.

3. Studies in which attempts are made to explain the fluctuations in residential building using multiple-correlation analysis, i.e., those of Roos,[14] Chawner,[15] and Tinbergen.[16] Of these studies only Tinbergen's second League of Nations volume has a paragraph on the long cycles, the explanation being confined, however, to an application of equation (1) without going into details.

REMARKS ON THE METHOD OF MULTIPLE CORRELATION

The present study, like the studies mentioned under group 3, makes use of the method of multiple correlation. In the writer's opinion this is the only way to break up the fluctuations in variables which are affected by a number of factors, acting simultaneously. It can be seen from computations quoted in the *Real Estate Analyst*, November, 1939, that there is a low correlation between residential building and the index of rents. Such results are meaningless, however, because they do not take into account that building activity is influenced by a number of factors, which for some time may act together and in other periods counteract each other. The same objection can be raised against conclusions drawn from a comparison of graphs on residential construction, interest rates, population growth, etc. (Newman). Though multiple-correlation analysis may occasionally give rise to serious difficulties, it is probably the only logical way of analyzing economic time series, the fluctuations of which have been influenced by several factors simultaneously.

Two main ideas underlie the method of multiple correlation. In the first place it offers a way of testing statistically hypotheses concerning the causation of the fluctuations in the economic variables studied. In the second place it makes possible the determination of quantitative relationships, whereas deductive reasoning can in most cases give us

[12] W. H. Newman, "The Building Industry and Business Cycles," *The Journal of Business of the University of Chicago*, July, 1935.

[13] C. D. Long, Jr., "Seventy Years of Building Cycles in Manhattan," *Review of Economic Statistics*, Vol. 18, 1936, p. 183; "Long Cycles in the Building Industry," *Quarterly Journal of Economics*, Vol. 53, May, 1939, p. 371.

[14] C. F. Roos, "Factors Influencing Residential Building," Chapter VI in *Dynamic Economics*, Bloomington, Indiana, 1934, p. 69.

[15] L. J. Chawner, *The Residential Building Process; An Analysis in Terms of Economic and Other Social Influences*, Industrial Committee of the National Resources Committee, Housing Monograph Series No. 1, Washington, 1939.

[16] J. Tinbergen, *Statistical Testing of Business-Cycle Theories*, Vol. I, Geneva, 1939, p. 90; Vol. II, p. 48, p. 56, p. 162.

only qualitative information. One of the minor objections raised against the method is the contention that it does not take into account changes in the structure of the economy: institutional and technological changes, changes in the consumption habits, etc. This objection, however, is certainly invalid, for gradual changes can in most cases be accounted for by the introduction of a trend factor and there exist also devices adequate for dealing with cases of abrupt changes. Another criticism is to the effect that, because of the close parallelism in many economic time series, the method is not unambiguous and consequently a high correlation would not prove very much but merely be a restatement of a hypothesis in terms of regression coefficients. Experience shows, however, that in most cases it is extremely difficult to reach very satisfactory correlations, especially if the analysis is carried out with the great care that is a requirement of the method. In many cases moreover there exist possibilities of checking the results, e.g., by comparing results for different geographical regions. Sometimes conclusions that may be drawn from the results are subject to further statistical or theoretical verification. One of the most convincing tests exists indubitably in extrapolating the relations found and comparing them with the observed values.

STATISTICAL DATA USED IN THE INVESTIGATION

The necessary statistics have been taken from published material. Though for a more detailed analysis more statistical information would be required, it is hardly possible for a private investigator to charge himself with the task of carrying out statistical inquiries. The principal statistical series used are the following:

1. Nonfarm residential building. Number of family dwelling units upon which construction was started annually, as estimated by the Department of Commerce. For years previous to 1915, see Chawner, p. 13. The weaknesses of this index are discussed by Newman, p. 7.

2. Index of rents, compiled by the National Industrial Conference Board. It is a very sensitive index, especially, it seems, for the years since 1933. No use has been made of the index of the Bureau of Labor Statistics which covers only the wage earners and lower salaried groups.

3. Index of building costs, *Engineering News Record*. The index dates back to 1903. For comments on this index, see Newman, p. 17. Its main drawback is that it does not include land cost and miscellaneous items (management, architects' fees, selling expenses, depreciation of equipment).

4. Residential vacancies in nonfarm areas, 1930–1937, Department of Commerce, Chawner, p. 16.

5. Foreclosures per 100,000 families, percentages above or below

normal, the *Real Estate Analyst*, Nov. 24, 1939. Published only in graphical form.

6. Interest rate on mortgages. A satisfactory index is not available. Some investigators use the index of long-term interest rates, based on the yields of 60 high-grade bonds, as reported by Standard Statistics Co. No information is available as to how far this index may be considered representative. Roy Wenzlick published an average rate of interest earned on the mortgages owned by the 110 largest life insurance companies. There is no close relationship between the fluctuations in this index and the index of Standard Statistics Co.

7. Ratio of the total number of nonfarm families to the total number of available housing units in nonfarm areas, 1900–1938, annual figures, January 1, Chawner, p. 16. The method of estimation is explained on page 2 of Chawner's monograph, where figures on the annual net increase in number of nonfarm families are given also.

8. Average income per nonfarm family, computed by the writer from the estimates on national income of the National Bureau of Economic Research and the National Industrial Conference Board and Chawner's figures on total number of nonfarm families.

FACTORS GOVERNING RESIDENTIAL CONSTRUCTION

The analysis consists of two parts. In the first place we shall investigate residential building and try to explain its fluctuations. As has been suggested before, we may expect that one or more of the explanatory factors introduced will have an endogenous character, which means that their fluctuations are, at least partly, determined by conditions in the housing market. An analysis of these relationships will complete our picture of the endogenous mechanism which governs the long cycles in residential construction.

Among the factors which determine residential building we may distinguish first a group of factors which represent the profitability of owning houses. This "incentive to build" depends on rental income after due allowance has been made for the occupancy ratio and for the costs of ownership. The costs of ownership include taxes, amortization, maintenance, and interest payments on mortgages. The last three items depend on building costs, the interest payments on mortgages and also on the interest rate.

In principle two ways are open to introduce this "incentive to build" into the correlation analysis. Following Tinbergen (Vol. I, p. 91) one may either combine rent, building costs, and interest payments on a priori considerations, or one may carry out a "free" calculation, that is, introduce each of the variables mentioned separately into the correlation analysis. The first method has the advantage that it limits the

number of independent variables in the correlation analysis; the second method is in so far superior as it allows the introduction of lags. Moreover, it may happen that the indices used are incomparable, e.g., as regards their degree of sensitiveness. This would not impair the "free" calculation if we may assume that the lack of comparability is restricted to the amplitudes of the indices studied.

Of course, the decisions of the prospective builders are based on certain expectations regarding the future. These expectations, however, and this is one of the characteristics of our method, will depend chiefly on what has happened in the present or in the recent past. Without this assumption it would not be possible to explain the fluctuations in a time series by the fluctuations in other time series.

The influence of the occupancy ratio may be twofold. First vacancies reduce gross rental income. This explains why some investigators habitually multiply the index of rents by the occupancy ratio before deducting the costs of ownership (cf., e.g., Roos). The correctness of this procedure may be doubted, as it is well known that new houses, which offer the latest comforts and technical conveniences, show a smaller vacancy percentage than houses of an older type. It has often been argued that the building industry (or the people who buy houses from builders) pays attention only to the vacancy percentage of new houses. This may constitute one of the reasons why residential construction may remain at a relatively high level for some time after the housing market begins to show signs of an approaching depression. In any case, this aspect of the influence of the occupancy ratio cannot be large, as in 1932 residential vacancies amounted to 8 per cent, whereas in the same year rents had fallen to about 60 per cent of their peak in 1924. A merely technical difficulty results from the circumstance that data on vacancies are not available for years previous to 1930. We may, however, assume that the vacancy percentage can be approximated rather closely by a linear combination of average nonfarm family income and the ratio of total number of nonfarm families to the total number of available dwelling units. Such a correlation has been established for the years 1930–1937 and the regression equation found extrapolated for previous years.

More important probably is another aspect of vacancies. A continuous increase in the number of unoccupied houses will without doubt affect the financial situation in real estate and, therefore, will affect the ease with which credits may be obtained for new building. Of course, the willingness of banks to grant credits will depend on the foreclosure rate, which, therefore, has always been regarded as an important indicator in forecasting building activity. The foreclosure rate plays a prominent role in Roos's analysis of residential building in Saint Louis.

It has been argued, however, that the foreclosure rate is a derived factor, in so far as it seems to be closely related to other variables governing the situation in the housing market. Tinbergen, e.g., has emphasized that the foreclosure rate in Saint Louis is correlated with the number of unoccupied houses some time before. A similar relationship holds probably also for the country as a whole, provided we take into account that nowadays a much higher foreclosure rate is considered as "normal" than in the years immediately after the war. This may be ascribed probably to the circumstance that the credit givers are at the present time prepared to take greater risks because of the abundance on the capital market. The figures on the foreclosure rate, which have been published by Roy Wenzlick in graphical form only, do not warrant very accurate conclusions. It seems probable that the foreclosure rate should, at least partly, be seen as the result of a cumulative effect, inasmuch as it depends not only on the number of vacancies some time before, but also on the number of years during which a depressed situation has been in existence. For these reasons the foreclosure rate was introduced as an independent variable besides the variables mentioned above and others that will be indicated below.

The result happened to be unsatisfactory, as the regression coefficient attached to the foreclosure rate appeared to be very low, the increase in the coefficient of multiple correlation was negligible, and the pattern of the residuals remained practically unaltered. Analogous results were experienced when the occupancy ratio, estimated by the method indicated above, was introduced into the correlation analysis.

The interest rate, as measured by the yield on bonds, or by Roy Wenzlick's figures on the average interest rate earned by 110 life insurance companies, showed very little influence, if any at all. Moreover, its inclusion did not modify the pattern of the residuals. The result is in accordance with a view held by a number of authors, who believe that the interest rate does not influence residential building materially, probably because its influence works in two directions.

A second group of factors may be indicated as the influence of the "acceleration principle," though this is often mentioned in connection only with the production of capital goods. Newman (p. 55), in particular, emphasizes the importance of this principle. He quotes J. M. Clark's statement that the essential characteristic for the application of the "principle" is the durability of the goods studied, whether producers' or consumers' goods. According to the "acceleration principle" variations in the rate of change of the demand for shelter will result in variations in residential building. If we may assume that at each level of national income there exists a definite need for shelter, residential building will be proportional to changes in national income,

other factors being held constant. It is probably more accurate to consider separately fluctuations of the number of families and fluctuations of the average family income, as it seems reasonable to expect that these two factors are different with respect to the intensity by which they affect residential building, whereas there may be also a difference in the lags involved. The influence of an increase in the number of families will not need much comment. Changes in the average family income will largely affect the extent to which "doubling up" of families takes place. Moreover, this factor will represent the demand of people who want to live in houses built according to their own design.

Some authors have emphasized the importance of a speculative factor. This factor would constitute an added stimulus to residential construction during booms and discourage it in times of depression. It is very possible that the rate of change in average family income which is, of course, closely related to the rate of change in general business activity, not only represents the influence of the acceleration principle, but also accounts, to some extent, for a speculative factor. With respect to the demand for replacement we make the assumption that it does not show marked fluctuations. It might be, of course, that in times of prosperity the rate of replacement is somewhat above the normal level and during depressions somewhat below. This influence could be measured by introducing the average family income into the correlation analysis. The result was, however, negative. The regression coefficient found was unacceptable, as it was very small and negative.

The results of this part of our analysis may be represented by the following regression equation:

$$(2) \qquad b^* = 16.3r - 8.56c_{-1} + 5.80(i_{+1} - i) + 0.44p_{-2} - 545.$$

The meaning of the symbols is as follows:

b^* = number of dwelling units, in thousands, upon which construction was started annually, as given by the above formula,

r = index of rents, 1914–1938 = 100,

c_{-1} = index of building costs, 1914–1938 = 100,

i = average nonfarm family income, 1914–1938 = 100,

p_{-2} = annual increase in number of families, nonfarm, in thousands.

Lags or leads have been indicated by subscripts. The time unit chosen is a year. The regression analysis covered the period 1914–1938. A graphical representation is given in Figure 1, where the dotted line b indicates actual residential construction and the solid line b^* residential construction according to the above formula. The explanatory factors have been represented separately.

We shall now give a more detailed discussion of the results. The explanatory factors may be taken together into two groups. The first

group, representing the influence of the "incentive to build," includes the factors determining the profitability of owning houses. Among these rents and building costs are the most prominent. As we have

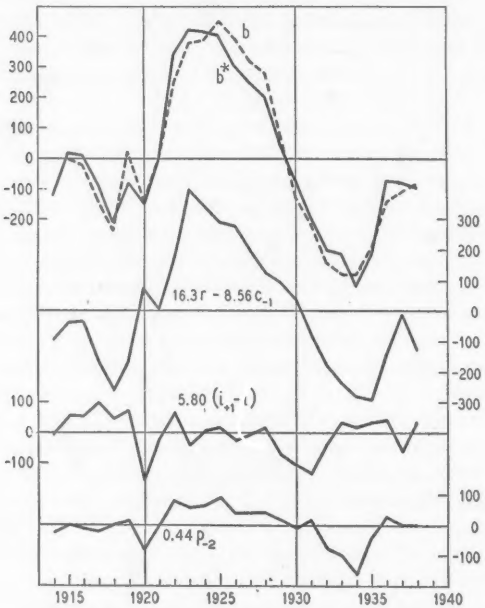

FIGURE 1.—Explanation of fluctuations in residential building, 1914–1938.

b, residential building, in thousands of dwelling units, estimates by Department of Commerce.

b^*, residential building, as computed from equation (2).

$16.3r - 8.56c_{-1}$, influence of "incentive to build."

r, index of rents, National Industrial Conference Board; 1914–1938 = 100.

c_{-1}, index of building costs, *Engineering News Record;* 1914–1938 = 100; lagged one year.

$5.80(i_{+1} - i)$, influence of changes in average nonfarm family income, i; index, 1914–1918 = 100.

$0.44p_{-2}$, influence of increase in number of nonfarm families, in thousands; lagged two years.

All variables are represented as deviations from their averages over 1914–1938.

seen before, the influence of the occupancy ratio and of the interest rate were negligible. Taxes have not been investigated separately. This factor may, probably with sufficient accuracy, be represented by a trend. An increasing trend in taxes would have to result in a declining

trend in residential building. No evidence has been found for the existence of this trend. Probably it is counterbalanced by some other trend, e.g., the influence of the continuous expansion of the metropolitan areas. The circumstance that nowadays mortgages cover a larger percentage of building costs may also have exerted some influence.

As both r and c have been represented as index numbers, averaging 100 over 1914–1938, it follows from the regression coefficients found that the influence of rents is about twice as large as that of building costs. As a matter of fact, the ratio is somewhat less than 2 as the building costs index does not cover certain items which, as was indicated before, account for about 20 per cent of total building costs. The result does not seem unreasonable if compared with a priori considerations (cf. e.g., Tinbergen's figures, Vol. I, p. 91).

Attempts have been made to establish the lags involved as accurately as possible. By using the method of distributed lags it was found that the lag which has to be attributed to the index of rents amounts to less than a month. As no great accuracy can be attached to this result, the lag has been put equal to zero in the above equation. Yet there is a small lag, as b refers to the number of dwelling units upon which construction was started. As the period of construction is rather short in the United States (four months according to Newman, p. 7), the lag between rents and building actually amounts to a few months.

The lag attached to the index of building costs deserves special attention. Here again use was made of the method of distributed lags. It was found that the "optimum" lag was between 11 months and a year, therefore a lag of a year was adopted. This result is not necessarily contradictory to current reports on the building industry, in which as a rule much attention is paid to the latest developments in building costs. It might quite well be that the more important decisions in the building industry are not so much based on the current fluctuations in building costs, but more on the underlying tread.

The second group of factors represents the influence of the "acceleration principle." The regression coefficient found for the influence of an increase in families seems to be reasonable as it is considerably below unity. (Both b and p are expressed in thousands of units.) The lag amounted to about two years, which seems rather large. It may probably be explained by the fact that the greater part of young families start on relatively small incomes and, therefore, cannot immediately exert a very effective demand. Of course, an increase in population will partly influence residential building indirectly through an increase in rents. An annual increase of 1 per cent in the average income will, according to the regression coefficient, effectuate an increase in new building of 5800 dwellings units, or 0.27 per cent of the average number of

available housing units in the period 1914–1938. This figure is too large to be considered as representing only the influence of changes in the degree of "doubling up." It seems probable, therefore, that this variable accounts also for a speculative factor, as suggested before. This is further confirmed by the lead found, as the fluctuations of those items of national income which chiefly determine the decisions of speculators will as a rule show a marked lead in comparison with the fluctuations in total income.

The question may be put as to how far the equation (2) is valid in times of depression. Of course, building cannot be negative, or more accurately, it cannot be below a negative figure, equaling annual replacement. Though in 1933 residential construction was still above zero, we should not overlook that equation (2) refers to the country as a whole. In certain regions of the country the situation might quite well have been so depressed that the regression equation would no longer have been valid.

The correlation cannot be considered very satisfactory, unless one takes into account certain restrictions mentioned before. The coefficient of multiple correlation amounted to 0.96. With respect to the discrepancies between actual and calculated residential building attention should, moreover, be paid to the following. As indicated in the Introduction, building activity was badly impaired during the years 1917–1918 and consequently jumped up in 1919. No special correction has been applied to allow for this disturbance. In 1923 and 1924 calculated building is considerably higher than the observed figures. This may be due to inaccuracies in the estimates of residential building. Throughout this study use has been made of the estimates on residential construction published by the Department of Commerce. These estimates are very close to the figures published by Wickens and Foster, except for 1923 and 1924, in which years the latter figures are higher. For the years 1925–1928 the fit is unsatisfactory, which is probably owing to the construction of apartments, showing for those years a boom reaching its turning point not until 1928, whereas total residential construction has been on the decline since 1925. More statistical information on the factors governing the construction of apartments would be required in order to make possible a more detailed investigation into this point of the analysis. With regard to the last five years covered by the regression equation, it has been remarked before that the index of rents is probably very sensitive for this period. Moreover, the estimates on the annual increase of the number of nonfarm families are for these years probably less reliable as they could no longer be based on interpolation. The figures on national income for the last two years are also not very accurate.

FACTORS GOVERNING FLUCTUATIONS IN RENTS

The next step of the analysis concerns the "independent" factors of equation (2). As indicated before, we shall have to determine which of these factors is itself governed by conditions in the housing market. As the building industry forms an important branch of the economy, it itself exerts some influence on the increase of population and also on the fluctuations of average income. For the present we shall, however, assume that this influence is negligible. With respect to building costs we shall make the same assumption, though in this case the hypothesis seems to be less justified. However, an investigation into the factors influencing the fluctuations of the index of building costs reveals that the influence of residential building is relatively unimportant, the main determining factors being total construction activity and the general business situation. We shall show later in how far our explanation of the long cycles in residential construction is affected by these simplifying hypotheses.

Thus we are left with the index of rents. Its fluctuations have been explained by Chawner (p. 5) and independently also by Tinbergen (Vol. II, p. 56). Our analysis differs from their results only in minor points.

Fluctuations in the level of rents will depend on demand and supply factors. As demand factors we introduce:

(a) The total number of nonfarm families as an indicator of the "normal" need for shelter.

(b) Average income of nonfarm families. On this factor depends the degree of "doubling up." Moreover, it affects the demand for houses in the higher rent brackets.

(c) Long-run fluctuations of the cost of living without rent. The assumption has been made that minor fluctuations in the cost-of-living index do not affect the demand for housing, as moving is inconvenient and costly and also because dwelling constitutes a "conspicuous" item of consumption. Therefore, this factor has for the period considered been represented by a trend, which has to be positive as the percentage of the total cost of living covered by rent has gradually increased.

As supply factor we introduce the total number of available housing units. In the regression analysis it has been combined with (a) into the ratio of total number of families to the total number of dwelling units.

By the method of least squares the following regression equation has been obtained:

$$r^* = 4.9n_{-2} + 0.521i_{-1} + 0.53t - 428, \tag{3}$$

in which:

r^* = index of rents, 1914–1938 = 100, as given by the above formula,
n = ratio of number of families to number of available dwelling units, percentage,
i = average income of nonfarm families, 1914–1938 = 100,
t = trend in years, 1926 = 0.

Much care has been given to an accurate determination of the lags involved. By introducing distributed lags it was found that the "optimum" lag for n amounted to approximately 2 years, taking into ac-

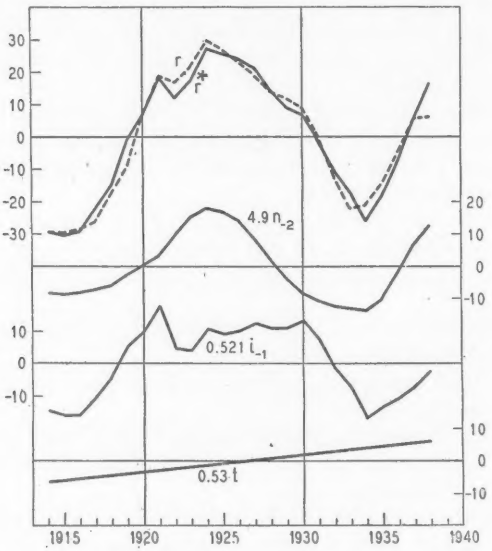

FIGURE 2.—Explanation of fluctuations, in rents, 1914–1938.
r, index of rents, National Industrial Conference Board; 1914–1938 = 100.
r^*, index of rents as computed from equation (3).
$4.9n_{-2}$, influence of ratio (percentage) of total number of nonfarm families to total number of available dwelling units; lagged two years.
$0.521i_{-1}$, influence of average nonfarm family income, i; index, 1914–1938 = 100; lagged one year.
$0.53t$, trend.

count that Chawner's figures refer to January 1. (Therefore 6 months had to be added to the lag found for Chawner's figures, which was a year and a half.) Our result is in accordance with Tinbergen's findings, whereas Chawner arrives at a lag of 9 months only. As in the United States houses are as a rule rented for a period of a year, our result does

not seem to be unreasonable, especially if one takes into account the stickiness of rents due to various imperfections in the housing market.

A graphical representation of equation (3) is given in Figure 2. The dotted line r refers to the actual figures, the solid line r^* indicates rents as estimated by the above equation (3).

The coefficient of multiple correlation amounted to 0.97. The closeness of fit is satisfactory as may be seen from Figure 2.

EXPLANATION OF THE LONG CYCLES IN RESIDENTIAL CONSTRUCTION

Equations (2) and (3) provide the necessary base for an explanation of the long cycles in residential building. Elimination of r leads to the following relationship:

$$b = 79.6n_{-2} + 8.50i_{-1} + 8.65t - 8.56c_{-1} + 5.80(i_{+1} - i) + 0.44p_{-2} - 7425. \tag{4}$$

According to this equation building activity is related to a shortage or oversupply of dwellings two years before. Assuming for a moment that all exogenous factors—average family income, increase in number of families, and index of building costs—are kept constant, we are left with an equation similar to equation (1). In order to make the two equations strictly comparable we have to take into account that n is a ratio, whereas in (1) both members have to be expressed in the same units. Denoting the shortage or oversupply of dwelling units, in thousands, by d, the relationship between n and d is as follows:

$$n = -0.0045 + \text{constant}. \tag{5}$$

This formula has been derived by making use of the fact that on the average over the period 1914–1938 the ratio n has been equal to 96.3 per cent, whereas the number of dwellings has averaged 21.4 million units.

From (4) and (5) it follows that, omitting all constants,

$$b = -0.36d_{-2}. \tag{6}$$

This equation is strictly analogous to (1), if we put $a=0.36$ and $\theta=2$ years.

Consequently, the constant B, mentioned in the discussion on equation (1), has the value 0.72. Comparing this result with the summary of the possible solutions of equation (1) given before, we see that in the case of (6) the solutions are damped oscillations.

We shall now investigate in how far the above results have been affected by the simplifying assumptions introduced in the course of the analysis. It can easily be shown that their effect is very small and does

not change the results materially. We have assumed before that, e.g., the building-costs index is not influenced by residential building. In order to investigate the validity of this hypothesis, the author has made an attempt to give an explanation of the fluctuations in the index of building costs. Among the factors which determine these fluctuations are total construction activity (of which residential construction is only a part) and the general business situation. For a detailed analysis it would be necessary to investigate separately the factors which determine the fluctuations of the wages in the building industry, etc. However, it is not necessary for the purposes of this article to carry out such an elaborate analysis, because it can be shown that the influence of residential building, which is the only factor in which we actually are interested here, is rather unimportant.

Between 1925 and 1933 the index of building costs (1915–1938 = 100) dropped from about 115 to about 85. Over the same period residential building shows a decline from about 900,000 to less than 100,000 dwelling units. Comparing these facts and assuming that the decline in residential construction accounts for 50 per cent of the decline in building costs, we get the following formula:

$$c = 0.018b + \text{other factors} + \text{constants}. \tag{7}$$

It has been assumed that there is no lag between the fluctuations in b and in c. Substituting (7) in (4) we get, instead of (6):

$$0.15b_{-1} + b = -0.36d_{-2}. \tag{6'}$$

This equation can easily be transformed into one of the form (1). Putting $\alpha = 0.15/1.15 = 0.13$, we may write with sufficient approximation:

$$1.15b_{-\alpha} = -0.36d_{-2} \tag{6''}$$

or

$$b = -0.31d_{-1.87}. \tag{6'''}$$

The solutions of (6''') too will show damped oscillations. Their period will be somewhat longer and the damping degree somewhat larger than in the case of (6).

It has been assumed above that half of the decline in building costs between 1925 and 1933 is due to the decline in residential construction. This is probably an overestimation of the influence of that factor. It is clear, therefore, that the results are modified only slightly by a refinement of the analysis with respect to the index of building costs.

Figure 3 gives a graphical representation of equation (4), extrapolated for the years prior to 1915. The extrapolation was not possible

in the case of equations (2) and (3) because no index of rents is available for the prewar period. It should be borne in mind that the validity of the extrapolation is somewhat doubtful, as, e.g., the index of building costs is for the years covered by the extrapolation considerably below the level which this index attains in 1915–1938, the period for which the regression coefficients were established.

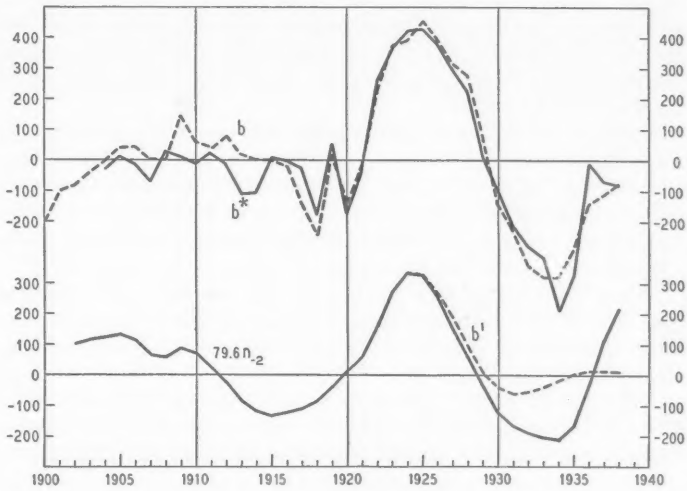

FIGURE 3.—Explanation of residential building after elimination of index of rents, 1903–1938.

b, residential building, in thousands of dwelling units, estimates of Department of Commerce.

b^*, residential building, as computed from equation (4), extrapolated for years prior to 1914.

$79.6n_{-2}$, influence of ratio of number of families to number of available dwelling units; lagged two years.

b^1, the endogenous cycle, starting from the situation in 1924–1925 (omitting all exogenous variables).

The first term of equation (4) has been represented separately in the lower part of Figure 3. The dotted line represents the endogenous cycle, which would develop from the actual situation in 1925, if it were possible to keep all exogenous variables constant. The period of the cycles is about 12 years. The oscillations are very damped: the cyclical fluctuations disappear practically after completion of one cycle.

The period found is shorter than the average length of the cycles as

observed during the last hundred years. The high damping degree is also remarkable in view of the regularity with which the cycles have appeared in the past. While considering these results, one should keep in mind that they have been derived from the observations over the postwar period. It might quite well be that before 1914 one or more of the lags involved was longer than they are now. If the lag in equation (6), which is a sum of various lags, formerly had been equal to $3\frac{1}{2}$ years instead of little more than 2 years now, the period of the solutions would have been about 15 years and the damping degree much less than was found above. This follows from the discussion of the solutions of equation (1). Under the assumption made the constant B would be equal to $0.36 \cdot 3.5 = 1.26$.

If the above analysis is right, it is to be expected that in the future the building cycles will be more irregular than they have been in the past, because residential building will depend to a greater extent on external disturbances and influences. The results will remain to be of value because they explain how residential building will develop from a situation of shortage or oversupply on the housing market.

Finally the question may be put as to how far the results obtained may be used for other purposes, e.g., forecasting. As we have seen some of the independent variables influence rents and residential building after a certain time lag.

This makes it possible to predict residential building one or two years ahead. In the writer's opinion, however, only rough conclusions can be reached in this respect. A more detailed analysis would be needed in order to obtain results that might be used for more accurate methods of forecasting residential building than are available now. Attention has been drawn before, e.g., to the construction of apartments. The assumption might be made that the construction of apartments is partly a saturation phenomenon. This would lead to an improvement of the correlation obtained in Figure 1. As no further proof could be given to justify the hypothesis, it has been left aside in the present analysis.

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A NEW METHOD OF TREND ELIMINATION

By M. Kalecki and B. Tew

I. THE ESSENTIALS OF THE METHOD

1. We try here to develop a new method of trend elimination for the case when a time series y subject to trend is supposed at the same time to be a function of another time series x (e.g., y being the productivity of labour in a certain industry and x the volume of output in it). Or, if the time influence and that of x are assumed to be additive, the problem may be stated as follows: We want to represent y as a sum of two components,

$$y = f(t) + g(x), \tag{1}$$

where $f(t)$ is the trend function having certain definite properties plus random fluctuations and $g(x)$ the "pure" functional relation between x and y which we purport to discover.

The problem is usually solved by assuming a definite shape for the trend function and g (e.g., that they are both linear) and applying thereafter the method of double correlation. The method which we present here differs from this procedure in that: (1) No definite shape of g is assumed but this shape is obtained as a result of computation; (2) Though it is necessary to assume for the purpose of calculation a definite shape of the trend function, it may be demonstrated that, in so far as this function fulfils certain fairly general conditions, its shape does not affect very much the solution obtained for g.

2. The leading idea of our method is very simple. Let us denote the successive time units, say years, of the period considered by 0, 1, 2, $\cdots$, n. Suppose that in the periods k and l the variable x has the same value or $x_k = x_l$. It follows then from the equation (1):

$$y_l - y_k = f(l) - f(k).$$

If we denote the first differences of the function f by Δ_1, Δ_2, $\cdots$, the last equation may be written

$$\sum_{r=k+1}^{r=l} \Delta_r = y_l - y_k.$$

If x is a fluctuating variable, we shall have many equations of this type (this is, however, subject to the qualification that in general x is not exactly equal in any two periods; the necessary adjustment will be considered in paragraph 3). Now taking Δ_1, Δ_2, $\cdots$, as unknowns we have a number of equations and thus we can express these first

differences of f in terms of a few of them, say Δ_a, Δ_b, and Δ_c. Thus we reach the point where we must make some assumptions about the shape of the trend function.

We assume that the differences of the function $f(t)$, Δ_1, Δ_2, $\cdots$, should deviate as little as possible from a constant which amounts to the function $f(t)$ having a possibly steady arithmetical rate of change. Now we obtain from the theorem of least squares the condition:

$$\sum \left(\Delta_r - \frac{\sum \Delta_s}{n} \right)^2$$

should be a minimum. Δ_r being expressed in terms of Δ_a, Δ_b, and Δ_c, the same is true of this sum. Thus it is easy to obtain from the above

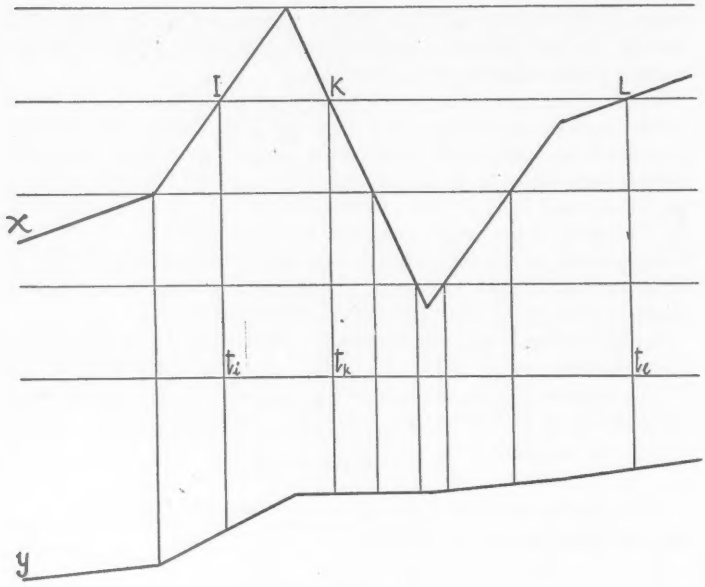

FIGURE 1.

condition the numerical values of Δ_a, Δ_b, and Δ_c and consequently also those of all Δ_r.

It must be mentioned that the condition of Δ_r deviating as little as possible from a constant is not equivalent to $f(t)$ deviating as little as

possible from a linear function of t. But the assumption that the rate of change of $f(t)$ should be as steady as possible is at least as reasonable as the latter one, and much more convenient for the purpose of calculation. It may be also shown that the difference in the function g obtained on either of these assumptions is small if the random component is not very great as compared with the deviations of $g(x)$ from the mean.

3. As already mentioned certain difficulties arise in the application of the above method since in general x is not exactly equal in any two years, but it may be adjusted as follows: Let us draw the time curve of x and under it that of y (see Figure 1; we obtain them by joining the "annual" points by straight lines). Let us find the points of intersection of the x curve with horizontals drawn at certain equal intervals, mark their abscissae, and project them on y curve. Let us denote the abscissae of these points by $t_0, t_1, t_2, \cdots, t_m$. If the points of intersection of a horizontal with the x curve are, say, I, K, L, their abscissae t_i, t_k, t_l, and the corresponding ordinates y_i, y_k, y_l, then, since x for these three points is equal, we have the equations:

$$f(t_k) - f(t_i) = y_k - y_i \quad \text{and} \quad f(t_l) - f(t_k) = y_l - y_k;$$

or, denoting the first differences of f by $\Delta_1, \Delta_2, \cdots$,

$$\sum_{r=i+1}^{r=k} \Delta_r = y_k - y_i \quad \text{and} \quad \sum_{r=k+1}^{r=l} \Delta_r = y_l - y_k.$$

Such equations can be obtained for each horizontal and then we are able to express all Δ's in terms of some of them, say, $\Delta_a, \Delta_b, \Delta_c$. And so it will be obviously possible to express in terms of them the rates of change of f, i.e., $\Delta_r/(t_r - t_{r-1})$.

Now we again minimize the sum of squared deviations of these rates of change from the average rate of change $\sum\Delta_s/(t_m - t_0)$. Since the intervals $t_{r+1} - t_r$ are in general not equal, the condition of least squares must now be written as follows:—

$$\sum (t_r - t_{r-1}) \left\{ \frac{\Delta_r}{t_r - t_{r-1}} - \frac{\sum \Delta_s}{t_m - t_0} \right\}^2 = \text{minimum}.$$

We obtain from this condition $\Delta_a, \Delta_b, \Delta_c$, and then all Δ_r.

4. After having determined the first differences of the function f we can now obtain this function itself as the cumulated first difference plus an arbitrary constant C. The function g may be then obtained by subtracting f from y. It contains, of course, also the constant C. This is not at all surprising since the existence of trend implies that the

shape of g must change in time. If, for example, y is the productivity of labour and x the output, the function g must change because of technical progress. If we define C from the condition:

$$y_1 = g(x_1),$$

where y_1 and x_1 are productivity and output in the first year, the function g corresponds to the technique of this year except for the random error which may be involved in y_1. We obtain the "average" g by imposing the condition

$$\sum y = \sum g(x).$$

The advantage of taking "average" g is its being less liable to random error than g derived from such an equation for a particular year.

Since we have $g(x)=y-f(t)$ the last equation may be also written:

$$\sum f(t) = 0. \tag{2}$$

II. APPLICATION OF THE ABOVE METHOD OF TREND ELIMINATION TO AN INDEX OF THE REAL LABOUR COST OF PRODUCTION IN THE BRITISH STEEL INDUSTRY

1. The two variables with which we are here concerned are:

y, the index of real labour cost, i.e., man-hours per unit of output,
x, the index of output,[1]

in the British Steel industry for the period 1920–35, the base year being 1930. (The years 1921 and 1926 were excluded, since they were years of great strikes.) These variables are also plotted in Figure 2, the annual values being connected linearly. We assume that y can be represented as the sum of two components

$$y = f(t) + g(x),$$

and our problem is to evaluate $g(x)$.

First we draw the horizontals,

$$\begin{aligned} x &= 120, \\ x &= 110, \\ x &= 100, \\ x &= 90, \\ x &= 80, \\ x &= 70, \end{aligned}$$

and drop perpendiculars on to y from the points of intersection of x with these horizontals. In all there proved to be 25 such perpendiculars.

[1] In this index the various types of steel products were weighted according to their estimated average wage cost.

The points at which these perpendiculars cut the time scale were labelled $t_0, t_1, t_2, \cdots, t_{24}$.

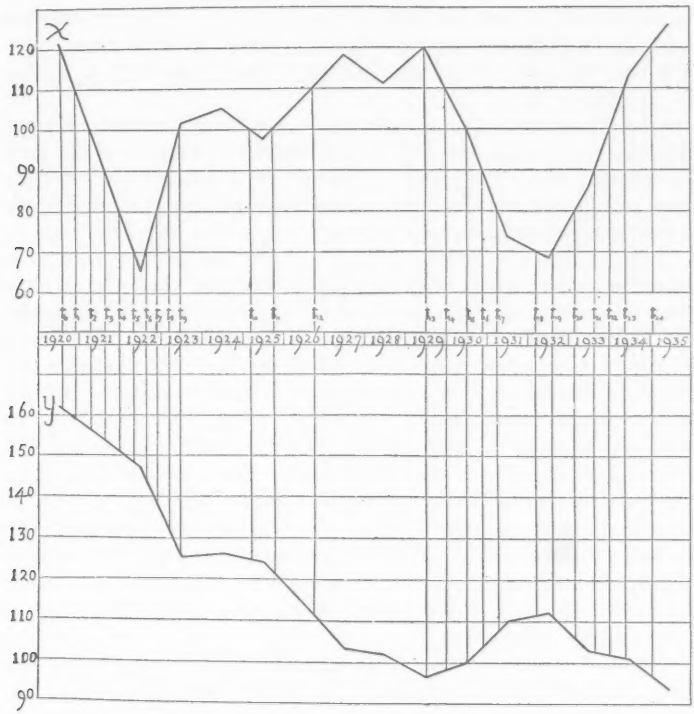

FIGURE 2.—x = output of the British steel industry; y = real labour cost; 1930 = 100.

The first differences of the unknown function $f(t)$, i.e., the changes in the value of $f(t)$ during the time intervals,

$$t_1 - t_0,$$
$$t_2 - t_1,$$
$$\cdots\cdots,$$
$$t_{24} - t_{23},$$

we denoted by $\Delta_1, \Delta_2, \Delta_3, \cdots, \Delta_{24}$, respectively.

Since the difference between the values of y at two dates at which x has the same value (e.g., t_0 and t_{13} in this example) is attributed solely

to the change in t, this difference is the sum of the Δ's lying between the two dates. Thus, in our example, x takes the same value, 120, at t_0 and t_{13}. At t_0 the value of y can be read off as the value taken by y at the foot of the first perpendicular, namely 161.5, while the value of y at t_{13}, read off in a similar manner, is 96.5. The difference (96.5–161.5 $= -65$) is the sum of the Δ's lying between t_0 and t_{13}: i.e., $\Delta_1+\Delta_2+\cdots\Delta_{13}=-65$.

Since, later on at t_{24}, x again takes the value 120, the value of $\Delta_{14}+\Delta_{15}+\cdots+\Delta_{24}$ can also be obtained; and in general on each horizontal ($x=120$, 110, 100, 90, 80, and 70) equations involving Δ's may be obtained one fewer in number than the number of times which x cuts the horizontal. In this example 19 equations were obtained.

2. The problem of obtaining all the Δ's in terms of a few of them could be tackled by solving the 19 equations as they stand, but we have found that by setting the equations in the manner described below the calculation is greatly facilitated, since it is then possible to express each of the Δ's in terms of only one of the "unknown" Δ's. Consider any "bay" in the x curve; in our example let us take for instance the one which extends from t_{13} to t_{24}. None of the horizontals which we consider pass through the very end of the bay, i.e., the minimum value of x in this period, since in 1932 x fell to 68, which is slightly lower than the lowest horizontal which we consider ($x=70$). Consequently one Δ, namely Δ_{19}, may be evaluated immediately, since the x curve cuts the horizontal $x=70$ at two successive t intervals (t_{18} and t_{19}). The value of Δ_{19} may be read off the graph of the y curve as 0 (since $y=111$ both at t_{18} and at t_{19}) or we can save time by referring to our 19 equations, one of which will be $\Delta_{19}=0$.

The next highest horizontal, $x=80$, cuts the bay at t_{17} and t_{20}, so that another of our 19 equations will give the sum of $\Delta_{18}+\Delta_{19}+\Delta_{20}$. We find that the equation is

$$\Delta_{18}+\Delta_{19}+\Delta_{20}=-2.$$

One of these Δ's, Δ_{19}, we have already evaluated, so that we can write

$$\Delta_{18}+0+\Delta_{20}=-2$$

or

$$\Delta_{18}+\Delta_{20}=-2$$

Again, the horizontal $x=90$ cuts the bay at t_{16} and t_{21}, so that a third equation will be of the form:

$\Delta_{17}+\Delta_{18}+\Delta_{19}+\Delta_{20}+\Delta_{21}=$ some definite value, in this case -1.5.

Since we know that $\Delta_{18}+\Delta_{19}+\Delta_{20}=-2$ we can write:

$$\begin{aligned}\Delta_{17}+\Delta_{21} &= -1.5-(-2)\\ &= +0.5.\end{aligned}$$

In the same way the other Δ's in the bay may be evaluated in pairs, and we therefore get:—

(a) $\Delta_{19} = 0,$
(b) $\Delta_{18}+\Delta_{20} = -2,$
(c) $\Delta_{17}+\Delta_{21} = +0.5,$
(d) $\Delta_{16}+\Delta_{22} = +3.5,$
(e) $\Delta_{15}+\Delta_{23} = +1,$
(f) $\Delta_{14}+\Delta_{24} = -3.$

Consider next the adjacent bay in the x curve, extending from t_6 to t_{13}. The end of this bay (the maximum value of x between t_6 and t_{13}) lies on the horizontal $x=120$, so that there is not in this case a Δ corresponding to the end of the bay which may immediately be evaluated singly. However, since $x=110$ cuts the bay at t_{12} and t_{14}, Δ_{13} and Δ_{14} can be evaluated. Actually $\Delta_{13}+\Delta_{14}=-13$.

Owing to the inflexion in the x curve between t_9 and t_{11} the next horizontal ($x=100$) cuts the x curve at three successive t's, viz., t_9, t_{10}, and t_{11}. Consequently Δ_{10} and Δ_{11} may be read off as -1 and -3.5 respectively. In addition, since this horizontal cuts the other side of the bay at t_{15}, we can evaluate $\Delta_{12}+\Delta_{13}+\Delta_{14}+\Delta_{15}$. Thus we get:

$$\Delta_{12}+(\Delta_{13}+\Delta_{14})+\Delta_{15} = -21.5,$$
i.e., $$\Delta_{12}+(-13)+\Delta_{15} = -21.5,$$
i.e., $$\Delta_{12}+\Delta_{15} = -8.5.$$

The other Δ's in the bay can in a like manner be evaluated in pairs. Thus we get:—

(g) $\Delta_{13}+\Delta_{14} = -13,$
(h) $\Delta_{12}+\Delta_{15} = -8.5,$
(i) $\Delta_{11} = -3.5,$
(j) $\Delta_{10} = -1,$
(k) $\Delta_{9}+\Delta_{16} = -2,$
(l) $\Delta_{8}+\Delta_{17} = -3,$
(m) $\Delta_{7}+\Delta_{18} = -2.$

In the third bay in the x curve, we can evaluate further Δ's, singly or in pairs:—

(n) $\Delta_6 = -4,$

(o) $\Delta_5 + \Delta_7 = -8.5,$

(p) $\Delta_4 + \Delta_8 = -9.5,$

(q) $\Delta_3 + \Delta_9 = -8.5,$

$\Delta_{10} = -1,$ } Also evaluated in the

$\Delta_{11} = -3.5,$ } preceding bay,

(r) $\Delta_2 + \Delta_{12} = -12.5,$

(s) $\Delta_1 + \Delta_{13} = -17.5.$

We have 19 equations and 24 Δ's, so that we can express all the Δ's in terms of five "unknown" Δ's. Since the 19 equations each contain either one or two Δ's, each of the Δ's may either be evaluated immediately, or be expressed in terms of *one* "unknown" Δ. Thus, taking Δ_{13}, Δ_{12}, Δ_9, Δ_8, and Δ_7 as the five unknown Δ's, we get:—

(A)
$\Delta_1 = -17.5 - \Delta_{13}$, from equation (s),
$\Delta_{13} = \Delta_{13}$,
$\Delta_{14} = -13 - \Delta_{13}$, from equation (g),
$\Delta_{24} = +10 + \Delta_{13}$, from equations (f) and (g).

(B)
$\Delta_2 = -12.5 - \Delta_{12}$, from equation (r),
$\Delta_{12} = \Delta_{12}$,
$\Delta_{15} = -8.5 - \Delta_{12}$, from equation (h),
$\Delta_{23} = +9.5 + \Delta_{12}$, from equations (h) and (e).

(C)
$\Delta_3 = -8.5 - \Delta_9$, from equation (q),
$\Delta_9 = \Delta_9$,
$\Delta_{16} = -2 - \Delta_9$, from equation (k),
$\Delta_{22} = +5.5 + \Delta_9$, from equations (k) and (d).

(D)
$\Delta_4 = -9.5 - \Delta_8$, from equation (p),
$\Delta_8 = \Delta_8$,
$\Delta_{17} = -3 - \Delta_8$, from equation (l),
$\Delta_{21} = +3.5 + \Delta_8$, from equations (l) and (c).

(E)
$\Delta_5 = -8.5 - \Delta_7$, from equation (o),
$\Delta_7 = \Delta_7$,
$\Delta_{18} = -2 - \Delta_7$, from equation (m),
$\Delta_{20} = + \Delta_7$, from equations (m) and (b).

(F)
$\Delta_6 = -4,$ from equation (n),
$\Delta_{10} = -1,$ from equation (j),
$\Delta_{11} = -3.5,$ from equation (i),
$\Delta_{19} = 0,$ from equation (a).

The above expressions are arranged in their proper order in Column IV of the Table 1.

TABLE 1

I	II	III	IV	V	VI	VII
Time interval	Length of interval in years	Reciprocal of previous column	Δ_r	$\frac{\Delta_r}{t_r - t_{r-1}} - \bar{\delta}$	Δ_r	$f(t)+c$ at the end of each interval
						0
t_1-t_0	0.3	3.3	$-17.5-\Delta_{13}$	$-53.9-3.3\Delta_{13}$	− 3.7	− 3.7
t_2-t_1	0.4	2.5	$-12.5-\Delta_{12}$	$-26.8-2.5\Delta_{12}$	− 3.5	− 7.2
t_3-t_2	0.3	3.3	$-8.5-\Delta_9$	$-23.9-3.3\Delta_9$	− 4.45	−11.65
t_4-t_3	0.4	2.5	$-9.5-\Delta_8$	$-19.6-2.5\Delta_8$	− 5.8	−17.45
t_5-t_4	0.35	2.9	$-8.5-\Delta_7$	$-20.0-2.9\Delta_7$	− 5.75	−23.2
t_6-t_5	0.3	3.3	− 4.0	− 8.9	− 4.0	−27.2
t_7-t_6	0.25	4.0	Δ_7	$+4.4+4.0\Delta_7$	− 2.75	−29.95
t_8-t_7	0.3	3.3	Δ_8	$+4.4+3.3\Delta_8$	− 3.7	−33.65
t_9-t_8	0.3	3.3	Δ_9	$+4.4+3.3\Delta_9$	− 4.05	−37.7
$t_{10}-t_9$	1.7	0.6	− 1.0	+ 3.8	− 1.0	−38.7
$t_{11}-t_{10}$	0.6	1.6	− 3.5	− 1.5	− 3.5	−42.2
$t_{12}-t_{11}$	1.0	1.0	Δ_{12}	$+4.4+\Delta_{12}$	− 9.0	−51.2
$t_{13}-t_{12}$	2.75	0.4	Δ_{13}	$+4.4+0.4\Delta_{13}$	−13.8	−65.0
$t_{14}-t_{13}$	0.5	2.0	$-13.0-\Delta_{13}$	$-21.6-2.0\Delta_{13}$	+ 0.8	−64.2
$t_{15}-t_{14}$	0.5	2.0	$-8.5-\Delta_{12}$	$-12.6-2.0\Delta_{12}$	+ 0.5	−63.7
$t_{16}-t_{15}$	0.4	2.5	$-2.0-\Delta_9$	$-0.6-2.5\Delta_9$	+ 2.05	−61.65
$t_{17}-t_{16}$	0.4	2.5	$-3.0-\Delta_8$	$-3.1-2.5\Delta_8$	+ 0.7	−60.95
$t_{18}-t_{17}$	0.9	1.1	$-2.0-\Delta_7$	$+2.2-1.1\Delta_7$	+ 0.75	−60.2
$t_{19}-t_{18}$	0.4	2.5	0	+ 4.4	0	−60.2
$t_{20}-t_{19}$	0.55	1.8	Δ_7	$+4.4+1.8\Delta_7$	− 2.75	−62.95
$t_{21}-t_{20}$	0.5	2.0	$+3.5+\Delta_8$	$+11.4+2.0\Delta_8$	− 0.2	−63.15
$t_{22}-t_{21}$	0.35	2.8	$+5.5+\Delta_9$	$+19.8+2.8\Delta_9$	+ 1.45	−61.7
$t_{23}-t_{22}$	0.4	2.5	$+9.5+\Delta_{12}$	$+24.8+2.5\Delta_{12}$	+ 0.5	−61.2
$t_{24}-t_{23}$	0.65	1.5	$+10.0+\Delta_{13}$	$+19.8+1.5\Delta_{13}$	− 3.8	−65.0
						Sum: −1073.7

3. We expressed the time intervals (t_1-t_0), (t_2-t_1), $\cdots$, $(t_{24}-t_{23})$, in years by reading the values from the graph. These are shown in Table 1, together with their reciprocals (Columns II and III).

We then evaluated:

$$\Delta_1 \div (t_1 - t_0),$$

$$\Delta_2 \div (t_2 - t_1),$$

$$\cdots\cdots,$$

$$\Delta_{24} \div (t_{24} - t_{23}).$$

These correspond to the average slope of $f(t)$ in the various time intervals.

Next we calculated, the average slope of $f(t)$ throughout the period t_0 to t_{24}, i.e.,

$$\bar{\delta} = \frac{\sum \Delta}{t_{24} - t_0} = -\frac{64}{14.5} = -4.4.$$

Thus we obtained the difference between the average slope of $f(t)$ in the various time intervals and the average slope throughout the period, i.e.,

$$\frac{\Delta_r}{t_r - t_{r-1}} - \bar{\delta},$$

which is shown in Column V in Table 1.

We then had to find the values of the unknown Δ's (Δ_7, Δ_8, Δ_9, Δ_{12}, and Δ_{13}) which minimized

$$\sum (t_r - t_{r-1}) \left\{ \frac{\Delta_r}{t_r - t_{r-1}} - \bar{\delta} \right\}^2.$$

Substituting these values for the unknown Δ's in Column IV we were able to evaluate all the Δ's (see Column VI).

4. We were now able, by cumulating the Δ's, viz.

$$\begin{aligned} &0, \\ &\Delta_1, \\ &\Delta_1 + \Delta_2, \\ &\Delta_1 + \Delta_2 + \Delta_3, \\ &\quad \cdots\cdots\cdots\cdots, \\ &\Delta_1 + \Delta_2 + \Delta_3 + \cdots + \Delta_{24}, \end{aligned}$$

to obtain $f(t)+C$ at the end of each interval,[2] (i.e., the trend function plus an arbitrary constant). We obtained this arbitrary constant C from the condition expressed by equation (2).

The sum of the cumulated Δ's above is -1073.7, so that since $\sum f(t)+25C=-1073.7$ and $\sum f(t)=0$, we get $25C=-1073.7$, i.e., $C=-43$.

Having evaluated C, we subtracted it from $f(t)+C$ and obtained $f(t)$. From the equation:—

[2] Column VII in Table 1.

$$y = f(t) + g(x)$$

we were able to calculate $g(x)$. Since equal values of $g(x)$ were associated with equal values of x, the result may be presented thus:

Value of x	*Value of g(x)*
120	118.5
110	119.5
100	120.5
90	122.5
80	126.0
70	138.5

These two series are plotted in Figure 3. Since $y = 100$ in 1930, $g(x)$ is the real labour cost of production at various outputs—the effect of trend being eliminated—expressed as a percentage of the actual real labour cost (y) in 1930.

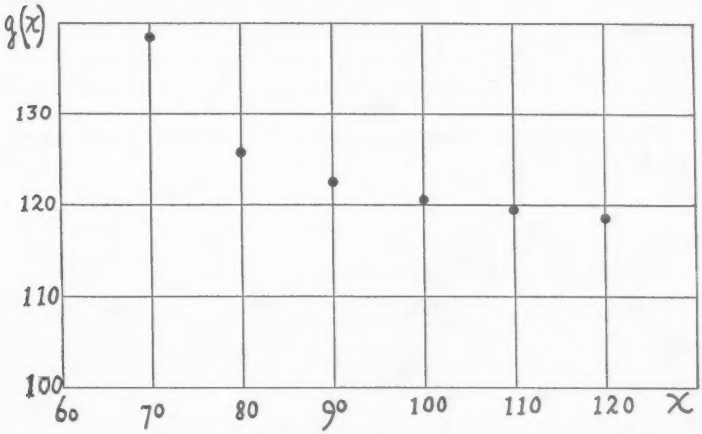

FIGURE 3.

III. APPENDIX

We will now show that the assumption we make about the rate of change of the trend function, if it fulfils certain requirements, does not greatly affect the function g obtained from the calculation. (To simplify the exposition we assume all $t_{\tau+1} - t_\tau$ time intervals to be equal.)

Let us suppose the rate of change of the trend function not to be constant but to fulfil some other condition. Let us denote by y', τ',

and g' the first differences of y, the trend function, and g respectively; and the deviations of y', τ', and g' from the mean by (y'), (τ'), and (g').

By means of our method (g') may be expressed in terms of some Δ's (say Δ_a, Δ_b, and Δ_c); it is easy to see that we shall have:

$$(g') = A\Delta_a + B\Delta_b + C\Delta_c,$$

where A, B, and C are determinate functions of x, and Δ_a, Δ_b, and Δ_c are constants to be obtained from the condition:

$$\sum [(y') - (\tau') - (g')]^2 = \text{minimum.} \tag{3}$$

Let us now determine, on the other hand, (g_1') corresponding to the assumption that the rate of change of the trend function is constant or $(\tau_1) = 0$; we have also here:

$$(g_1') = A\Delta_a + B\Delta_b + C\Delta_c,$$

where A, B, and C are the same functions of x as above while the values of Δ_a, Δ_b, and Δ_c are different and are determined by the condition:

$$\sum [(y') - (g_1')]^2 = \text{minimum.} \tag{4}$$

It follows that the difference between (g') and (g_1') may be represented as:

$$(g') - (g_1') = A\alpha + B\beta + C\gamma, \tag{5}$$

where α, β, and γ are constants.

Now from equation (3) we obtain:

$$\sum [(y') - (\tau') - (g')]A = 0, \qquad \sum [(y') - (\tau') - (g')]B = 0,$$
$$\sum [(y') - (\tau') - (g')]C = 0;$$

while from the equation (4):

$$\sum [(y') - (g_1')]A = 0, \qquad \sum [(y') - (g_1')]B = 0,$$
$$\sum [(y') - (g_1')]C = 0;$$

and, by subtracting:

$$\sum [(\tau') + (g') - (g_1)']A = 0, \qquad \sum [(\tau') + (g') - (g_1)']B = 0,$$
$$\sum [(\tau') + (g') - (g_1)']C = 0.$$

Now taking into account the equation (5) we have:

$$\sum [(\tau') + (g') - (g_1)'][(g') - (g_1')] = 0;$$

or:

$$\sum [(g') - (g_1')]^2 = - \sum (\tau')[(g') - (g_1')].$$

Dividing both sides by $\sqrt{\sum [(g') - (g_1')]^2}\ \sqrt{\sum (\tau')^2}$ we obtain

$$\frac{\sqrt{\sum [(g') - (g_1')]^2}}{\sqrt{\sum (\tau')^2}} = |\rho|, \tag{6}$$

where ρ is the correlation coefficient between $(g') - (g_1')$ and τ'. Since $|\rho| < 1$, we have:

$$\sqrt{\sum [(g') - (g_1')]^2} < \sqrt{\sum (\tau')^2}.$$

Now x is fluctuating in time (otherwise all our method would be inapplicable) and so will $g(x)$ be almost always. On the other hand our choice of the shape of trend function τ is in general such that (τ') does not fluctuate and very often it changes within a much shorter range than (g'). Under these conditions $\sum [(g') - (g_1')]^2$ will be small as compared with $\sum (g')$,[2] which means that g_1 corresponding to the assumption of a constant rate of change of the trend function does not differ much from g obtained on some other assumption about this function. If, however, there is an important change in the rate of trend while the fluctuations of $g(x)$ are relatively slight, the difference may prove substantial.

Cambridge

A DYNAMIC PROBLEM IN DUOPOLY

By A. Smithies and L. J. Savage

In 1925, Dr. C. F. Roos applied the calculus of variations to the problem of competition,[1] and, in the first part of his paper, solved a problem closely analogous to the Cournot case, where the production of one producer was assumed to be in a certain sense independent of that of the other. The purpose of the present paper[2] is to extend Dr. Roos's methods to cover problems made manageable by the use of difference and mixed difference and differential equations.[3] We shall work with a highly simplified model constructed for its analytical possibilities rather than its reflection of reality. While the conclusions are, we think, not without realistic significance, we regard this paper as chiefly of methodological importance.

In Part A we shall attempt to state the problem and the conclusions of the mathematical analysis, as far as is possible, in nonmathematical language. In Part B the rigorous mathematical argument will be developed. At the risk of some repetition, it seems to us that the purposes of each part can be best achieved by making them, as far as possible, independent of each other.

PART A

Suppose there are two duopolists faced with a demand function of "normal" shape (i.e., continuous and monotonically decreasing), and an identical cost function involving increasing or constant marginal costs. Suppose further that each of the producers must plan his output one unit of time (with appropriate choice of units) in advance. Then, in planning this output for any time, each producer will have to estimate his competitor's output at that time, on the basis of data available to him at the present time. Ideally, the whole past history of his own and his competitor's operations should be accepted as relevant evidence; but for the sake of simplicity we shall assume that one producer uses as

[1] "A Mathematical Theory of Competition," *American Journal of Mathematics*, 1925, p. 163.

[2] We are very grateful to Professor Rainich for reading the manuscript of this paper and for criticism and advice.

[3] See especially Frisch and Holme, "The Characteristic Solutions of a Mixed Difference and Differential Equation," Econometrica, Vol. 3, 1935, p. 225; Kalecki, "A Macrodynamic Theory of Business Cycles," Econometrica, Vol. 3, 1935, p. 327; Tinbergen, "Annual Survey: Suggestions on Quantitative Business Cycle Theory," Econometrica, Vol. 3, 1935, p. 241, at p. 276 and other references; James and Belz, "The Significance of the Characteristic Solutions of Mixed Difference and Differential Equations," Econometrica, Vol. 6, 1938, p. 326. Although the work of these authors has but slight direct application to our problem, we are greatly indebted to them indirectly.

the basis of his estimates his own and his competitor's present levels of output, and whether his competitor's output is at present increasing or decreasing over time.

The problem we wish to analyze is the process by which the two competitors adapt themselves to new situations. To be more specific; suppose that there has just passed a period over which the demand function has been shifting. During this period the competitors will have adapted themselves as best they can to changing conditions, but probably will not have been able to follow any coherent plan designed to maximize profits. But now demand conditions have become stable once more, and we wish to examine the equilibrium (if any) that will be achieved and the path over time by which the competitors will arrive at that situation. Clearly they will be interested not only in their profits in the equilibrium situation (which, in fact, may never be achieved since new disturbances may occur before adaptation is complete), but also in their profits during the period of adaptation. In planning his production for any unit period, a producer will therefore consider both the profits he makes in that period and also whether he is placing himself in a strategic position to make profits in future periods. In short, each competitor will aim at maximizing his profits over a period of time, and the question arises: what period? The answer is the period over which they expect demand conditions to remain stable. If this period is limited in length, it is easy to see that full equilibrium may not have been achieved by the end of it—since the sacrifices of present advantage involved in making the adjustment so rapidly may have been too great to be worth while. On the other hand, if the competitors expect conditions to remain stable indefinitely, we may expect that, if an equilibrium position exists, it will ultimately be achieved.

In the first unit period of time with which we are concerned the data on which each producer makes his estimates will be the outputs during the last unit of time in the period of change and confusion. In the second unit of time the outputs during the first unit of time will form the basis of estimation, and so on.

We shall now consider the behavior of the competitors over time in the following two cases: (1) Each competitor assumes that his rival's output, one unit of time from the present, will be a weighted arithmetic mean of their present levels of output. This is a generalization of which the Cournot assumption that one competitor assumes that his rival will maintain a constant output is an extreme special case. At the other extreme one producer will assume that his rival will always produce an output equal to his own. The case we shall investigate is the general one which includes these extremes and any case intermediate between

them. (2) Each competitor assumes that his rival's output will increase for the next unit of time at its present rate. Thus, if one producer's output is increasing at present at 10 per cent per month, it is assumed by his competitor that his output one month from now will be 10 per cent higher than at present.

Case (1). In this case a position of equilibrium always exists and will be attained if sufficient time is allowed to elapse. If the Cournot assumption is made, Cournot's solution will be the solution to the present problem. On the other hand, if it is assumed that outputs will be equalized, the solution will be the same as if the two competitors acted as a single monopolist.

Next, as to the path by which equilibrium is achieved. Clearly one competitor is prepared to sell more, the smaller the quantity his rival is selling. Hence, if he estimates that in the next period his rival is going to sell a smaller-than-equilibrium output, he will (except in extreme cases) plan to sell a greater-than-equilibrium output. If both competitors make such plans for the first unit period, they will find each other both selling greater-than-equilibrium outputs. They will then both try to contract production in order to restore prices, and will move to less-than-equilibrium outputs in the second period, and so on. Output for each competitor will behave cyclically over time and the length of the cycles will be about two units of time.

When a competitor contemplates a greater-than-equilibrium output, his inclinations will be checked by the prospect of subsequently having to contract. But this prospect will be less bleak if he thinks his rival will also contract rather than maintain a high level of output. Hence, the greater the extent to which the Cournot assumption is made, the more the cycles will be damped. But they will be damped in any case, and eventually stable equilibrium will be reached.

The amplitude of the cycles and their phase will depend largely on the initial conditions. If the rates of output during the initial interval are both greater or less than their equilibrium values, the cycles will have a greater amplitude than if the initial output of one competitor is greater and of the other less than its equilibrium value. In the latter case, the cycles may be of different phase and each will have a damping effect on the other. It may happen that the initial conditions are such that the estimated outputs are each equal to equilibrium output, and in this case equilibrium will be attained immediately.

Case (2). The second case yields cyclical solutions of similar character to those described above, but, as is fairly evident, we cannot say that stable equilibrium is achieved for all initial conditions. We can say, however, that if the initial outputs are approximately linear over time, the cycles will be damped over a finite period of time and output

for each competitor will tend towards the value given by the Cournot solution. If the initial outputs are in fact linear functions of time, the cycles will be damped over any period. We shall argue, further, that the "explosive" cases are in general cases where it is unreasonable to suppose that linear extrapolation would be used as the method of estimation.

The importance of the cycles from the point of view of several economic fluctuations depends on their period. In many industrial processes this is too short for the cycles to be of great interest. In such cases, we prefer to stress not the cycles but the tendency towards equilibrium which our analysis reveals.

PART B: MATHEMATICAL

I

We shall confine ourselves to considering two identical producers operating in a perfect market for a single commodity. This implies that they are confronted with the same (unchanging) demand function; that their cost functions are identical; that there is a single price ruling in the market; and that their reactions to each other's behavior are the same.[4]

Suppose that production at time t must be planned at time $t-1$ (with appropriate choice of units of time); and that production at t is also sold at that time. Then in planning his output each producer must conjecture what his competitor will be producing one unit of time in advance. If these conjectures are based on economic quantities at all, it seems reasonable to assume that the quantities that will be taken into account by each producer are his own and his competitors present levels of output and their rates of change. Thus, letting $f(t)$ and $\phi(t)$ denote the rates of production at time t, and $f^*(t)$ and $\phi^*(t)$ the outputs of each producer as conjectured by the other, we can write

$$f^*(t) = F_1\big[f(t-1), \phi(t-1), f'(t-1), \phi'(t-1)\big],$$

and similarly for $\phi^*(t)$, yielding a function F_2, analogous with F_1. We shall be concerned with the following two cases:[5]

(1) Each producer at time $t-1$ assumes that his competitor's output at time t will be a weighted arithmetic mean of their present levels of output, i.e.,

[4] It will be seen that the analysis can be pushed a considerable distance in the perfectly general case, but that in that case it is difficult to obtain significant qualitative results.

[5] We shall not consider cases where $\phi'(t-1)$ appears as a variable in F_1. Although the method to be followed is clear, to find a solution for particular functions is a matter of great complexity.

(1a) $$f^*(t) = mf(t-1) + n\phi(t-1),$$

(1b) $$\phi^*(t) = m\phi(t-1) + nf(t-1).$$

(2) Each producer assumes that his rival's output will continue to change at its present rate of change over the next unit of time, i.e.,

(2a) $$f^*(t) = f(t-1) + f'(t-1),$$

(2b) $$\phi^*(t) = \phi(t-1) + \phi'(t-1).$$

A combination of these hypotheses is probably more realistic than either of them alone, but for the sake of simplicity it seems preferable to deal with them separately. This will be done in Sections II and III.

Let $p(t)$ be the price at time t, and let the demand function be linear, of the form,

(3) $$p(t) = a[f(t) + \phi(t)] + b, \qquad a < 0, b > 0.$$

Let total costs for each producer be of the form:

(4) $$C[f(t)] = \tfrac{1}{2}cf(t)^2 + df(t), \ c > 0, d > 0, \text{ and } d < b.$$

In planning his production, each producer will have to work on the basis of an estimated price, which will depend on his own planned output and his estimate of his rival's output. Let $p_1^*(t)$ and $p_2^*(t)$ be the prices at time t as estimated by the producers. Thus from (1) and (3),

(5a) $$p_1^*(t) = a[f(t) + \phi^*(t)] + b,$$

(5b) $$p_2^*(t) = a[\phi^*(t) + f(t)] + b.$$

Each producer will then attempt to maximize his estimated profits over a period, 0 to N. The profits which the competitors will attempt to maximize will be given by

(6a) $$\pi_1 = \int_0^N [p_1^* f - C(f)]dt,$$

(6b) $$\pi_2 = \int_0^N [p_2^* \phi - C(\phi)]dt,$$

where the limits N and 0 indicate that the integration is to be performed over N time intervals, extending from 1 to N.[6]

The shapes of f and ϕ which will maximize π_1 and π_2 will be the solution of simultaneous difference equations in problem (1) and mixed difference and differential equations in problem (2), and our results

[6] Profits should be discounted and/or accumulated to some particular point of time, but, for present purposes, no essential principle is obscured, and the analysis is simplified by omitting the discounting factor.

will depend on given values of f and ϕ in the range $(-1, 0)$, which are assumed bounded and integrable. Following James and Belz,[7] these values will be termed "the constraints."

II

Case (1). Let $\bar{f}(t)$ and $\bar{\phi}(t)$ be the optimum shapes of f and ϕ and let

$$f = \bar{f} + \omega_1\psi, \tag{7a}$$

$$\phi = \bar{\phi} + \omega_2\theta, \tag{7b}$$

where ψ and θ are arbitrary functions of t, and ω_1 and ω_2 are parameters, $0 < \omega_1, \omega_2 < 1$.

The functions f and ϕ, $\bar{f}$ and $\bar{\phi}$ are assumed bounded and piecewise continuous. ψ and θ, therefore, also fulfill these continuity conditions.

Replacing f and ϕ in (6) by their values as given by (7), we can represent π_1 and π_2 as functions of ω_1 and ω_2 respectively. Since their points of discontinuity are finite in number, f and ϕ are integrable, and π_1 and π_2 are continuous and differentiable in ω_1 and ω_2 respectively.

Hence, since $f = \bar{f}$ where $\omega_1 = 0$ and $\phi = \bar{\phi}$ where $\omega_2 = 0$, the necessary and sufficient conditions that $\bar{f}$ and $\bar{\phi}$ may be the shapes of the functions that maximize π_1 and π_2 are:

$$\text{(a)} \qquad \frac{\partial\pi_1}{\partial\omega_1} = \frac{\partial\pi_2}{\partial\omega_2} = 0, \qquad \omega_1 = \omega_2 = 0,$$

$$\text{(b)} \qquad \frac{\partial\pi_1}{\partial\omega_1} < 0, \qquad \omega_1 \neq 0,\ \omega_2 = 0,$$

$$\frac{\partial\pi_2}{\partial\omega_2} < 0, \qquad \omega_2 \neq 0,\ \omega_1 = 0.$$

Differentiating (6a) with respect to ω_1 we have

$$\frac{\partial\pi_1}{\partial\omega_1} = \int_0^N \left[\frac{\partial f}{\partial\omega_1} p_1^* + f\frac{\partial p_1^*}{\partial\omega_1} - C(f)\frac{\partial f}{\partial\omega_1}\right] dt.$$

Let us adopt a subscript notation to indicate lagged terms: for example,

$$f(t-1) = f_{-1}, \qquad f(t+1) = f_{+1}, \text{ etc.}$$

Then substituting from (4), (5a), and (7a) and collecting terms:

$$\frac{\partial\pi_1}{\partial\omega_1} = \int_0^N \{(2a-c)(\bar{f}+\omega_1\psi) + an(\bar{f}+\omega_1\psi_{-1}) + am(\bar{\phi}_{-1}+\omega_2\theta_{-1}) + (b-d)]\psi + (\bar{f}+\omega_1\psi)an\psi_{-1}\}\, dt. \tag{8}$$

[7] *Op. cit.*, p. 327.

Hence for condition (a) we have

$$\left(\frac{\partial \pi_1}{\partial \omega_1}\right)_{\omega_1=\omega_2=0} = \int_0^N \{[(2a-c)\bar{f} + an\bar{f}_{-1} + \bar{a}m\phi_{-1} + b - d]\psi + an\bar{f}\psi_{-1}\}\,dt = 0. \tag{9}$$

Since the constraint is given, $\psi \equiv 0$ in the range $(-1,0)$, or, which is the same thing, $\psi_{-1} \equiv 0$ in the range $(0, 1)$, and hence we may write

$$\int_0^N f\psi_{-1}dt = \int_0^{N-1} f_{+1}\psi dt.$$

Hence (9) may be rewritten:

$$\left(\frac{\partial \pi_1}{\partial \omega_1}\right)_{\omega_1=\omega_2=0} \int_{N-1}^N [(2a-c)\bar{f} + an\bar{f}_{-1} + am\bar{\phi}_{-1} + b - d]\psi dt + \int_0^{N-1} [an\bar{f}_{+1} + (2a-c)\bar{f} + an\bar{f}_{-1} + am\bar{\phi}_{-1} + b - d]\psi dt = 0. \tag{10}$$

Since ψ is arbitrary, the integrand of (10) must be zero, hence condition (a) involves:[8]

$$(2a-c)\bar{f} + an\bar{f}_{-1} + am\bar{\phi}_{-1} + b - d = 0, \tag{11a}$$

in the range $(N-1, N)$;

$$an\bar{f}_{+1} + (2a-c)\bar{f} + an\bar{f}_{-1} + am\bar{\phi}_{-1} + b - d = 0, \tag{11b}$$

in the range $(0, N-1)$.

If we follow the same procedure for maximizing π_2, we obtain conditions precisely analogous to those for π_1, namely

$$(2a-c)\bar{\phi} + an\bar{\phi}_{-1} + am\bar{f}_{-1} + b - d = 0, \tag{12a}$$

in the range $(N-1, N)$;

[8] As far as we can ascertain, this type of calculus of variations problem, is new. It is worth noting that the technique can be applied to the integral of any function of the form $F(f, f_{-1}, f_{-2}, \cdots)$. In the particular case $I = \int_0^N F(f, f_{-1})\,dt$ the conditions corresponding to (11a) and (11b) are as follows:

$$\frac{\partial F}{\partial f_{-1}} = 0, \text{ in the range } (N-1, N),$$

$$\frac{\partial F}{\partial f} = \left(\frac{\partial F}{\partial f_{-1}}\right)_{-1} = 0, \text{ in the range } (0, N-1).$$

$$\text{(12b)} \qquad an\bar{\phi}_{+1} + (2a - c)\bar{\phi} + an\bar{\phi}_{-1} + am\bar{f}_{-1} + b - d = 0,$$

in the range $(0, N - 1)$.

It is worth while to obtain analytic solutions to the problem. The general solution can then be obtained by combining these analytic solutions so as both to satisfy the conditions (11a) and (12a), and to fit the constraints.

The general solutions[9] of (11b) and (12b) may be obtained by solving the homogeneous equations obtained by omitting the constant terms, $b-d$, from those equations, and adding to those solutions the constant solution of (11b) and (12b).

If we eliminate ϕ from the homogeneous equations corresponding to (11b) and (12b), we obtain a fourth-order difference equation involving $\bar{f}$ alone, namely,

$$\text{(13)} \qquad \begin{aligned} &a^2n^2\bar{f}_{+4} + 2an(2a - c)\bar{f}_{+3} + (2a^2n^2 + (2a - c)^2\bar{f}_{+2} \\ &\quad + [a^2n^2 + an(2a - c)]\bar{f}_{+1} + (a^2n^2 + a^2m^2)\bar{f} = 0. \end{aligned}$$

Analytic solutions are found by putting $\bar{f}=y^t$ which leads to the following equation for y:

$$any^2 + (2a - c)y + a(n \pm m) = 0,$$

$$\text{(14)} \qquad y = \frac{-(2a - c) \pm \sqrt{(2a - c)^2 - 4a^2n(n \pm m)}}{2an}.$$

This will, in general, yield four solutions for y, whose characteristics are as follows:

(a) Where $(m+n)$ occurs in the discriminant: we have two solutions, y_1 and y_2, such that $y_1<0$, $y_2<0$; and $|y_1|>1$, $|y_2|<1$. The corresponding solutions to (13) will therefore be expocyclic, of the form:

$$\begin{aligned} y_1^t &= |y_1^t|\, e^{(2r+1)\pi it}, \\ y_2^t &= |y_2^t|\, e^{(2r+1)\pi it}, \end{aligned} \qquad r = 0, 1, 2, \cdots, \infty,$$

with y_1^t explosive and y_2^t damped.

(b) Where $(n-m)$ occurs in the discriminant:

(1) Where $n>m$: This case is analogous to (a), and we have,

$$\begin{aligned} y_3^t &= |y_3^t|\, e^{(2r+1)\pi it}, \\ y_4^t &= |y_4^t|\, e^{(2r+1)\pi it}, \end{aligned} \qquad r = 0, 1, 2, \cdots, \infty,$$

with y_3^t explosive and y_4^t damped.

[9] It is worth noting that up to this stage of the analysis there has been no need for the special assumption that the two producers are identical. But to carry the analysis further in the most general case involved considerable algebraic difficulties. However, it is readily seen that the analysis that follows may be very easily adapted to allow different coefficients of the linear terms, in the cost function for each competitor.

(2) Where $n<m$: We have $0<y_4<1$ so that $y_4{}^t$ will be a "damped" real exponential function.

If we eliminate $\bar{f}$ from the homogeneous equations, we obtain a fourth-order difference equation in $\bar{\phi}$ identical with (13), yielding four solutions, identical with those for $\bar{f}$.

It is clear that the expocyclic solutions are all of period 2, multiplied, except $r=0$, by periodic functions of period 1. That implies that the undertone and the overtones[10] corresponding to any solution of (14) are not "periodically independent."[11] By selecting a member of each set of solutions, we obtain four periodically independent solutions which are sufficient to represent the complete solution of (13). Obviously, it is most convenient to select the undertones. Thus

$$\bar{f}_H = \alpha y_1{}^t + \beta y_2{}^t + \gamma y_3{}^t + \delta y_4{}^t,$$

and analogously

$$\bar{\phi}_H = \alpha' y_1{}^t + \beta' y_2{}^t + \gamma' y_3{}^t + \delta' y_4{}^t,$$

where the coefficients α, α', β, β', etc., are arbitrary (complex) periodic functions of period 1. But these solutions must satisfy the homogeneous equations simultaneously. For this to be so, it will be necessary and sufficient that

$$\alpha = -\alpha', \qquad \gamma = \gamma',$$
$$\beta = -\beta', \qquad \delta = \delta'.$$

We must next consider the constant solutions of (11b) and (12b). These are equal to each other, and denoting their value by K we have

$$K = -\frac{b-d}{an+3a-c}, \qquad K>0.$$

The general solutions of (11a) and (12a) are, therefore:

$$\bar{f} = \alpha y_1{}^t + \beta y_2{}^t + \gamma y_3{}^t + \delta y_4{}^t + K, \tag{15}$$

$$\bar{\phi} = -\alpha y_1{}^t - \beta y_2{}^t + \gamma y_3{}^t + \delta y_4{}^t + K. \tag{16}$$

The next step is to determine the necessary and sufficient condition for (15) and (16) to satisfy the terminal conditions (11a) and (12a). By direct substitution, and applying (14), we can easily establish the following conditions for the range $(N-1, N)$:

$$\alpha = -\frac{\beta y_2{}^{t+1}}{y_1{}^{t+1}}, \tag{17}$$

[10] Cf. James and Belz, *op. cit.*, p. 330.

[11] We define periodic independence as the analogue of linear independence. Thus the f_r are said to be periodically independent if $\sum_r \alpha_r f_r = 0$ when and only when all the $\alpha_r = 0$, the α_r being periodic functions of period 1.

$$\gamma = -\frac{K + \delta y_4^{t+1}}{y_3^{t+1}}. \tag{18}$$

In order to complete the solution to our problem, it now remains to determine the values of the coefficients which will make the general solutions, subject to (17) and (18), fit the constraints over the range $(-1, 0)$. Let the values of f and ϕ in that interval be f_0 and ϕ_0 respectively. Substituting τ for t in the range $(-1, 0)$, we must then have:

$$\alpha y_1^{\tau} + \beta y_2^{\tau} + \gamma y_3^{\tau} + \delta y_4^{\tau} + K = f_0, \tag{19}$$

$$-\alpha y_1^{\tau} - \beta y_2^{\tau} + \gamma y_3^{\tau} + \delta y_4^{\tau} + K = \phi_0. \tag{20}$$

By making the substitution $\tau+N+1=t$, (17) and (18) can be expressed in terms of τ. Thus:

$$\alpha = -\frac{\beta y_2^{\tau+N+1}}{y_1^{\tau+N+1}}, \tag{21}$$

$$\gamma = -\frac{K + \delta y_4^{\tau+N+1}}{y_3^{\tau+N+1}}. \tag{22}$$

Equations (19)–(22) can now be solved and we have

$$\alpha = \frac{f_0 - \phi_0}{2y_1^{\tau}\left[1 - \left(\frac{y_1}{y_2}\right)^{N+1}\right]},$$

$$\beta = \frac{f_0 - \phi_0}{2y_2^{\tau}\left[1 - \left(\frac{y_2}{y_1}\right)^{N+1}\right]},$$

$$\gamma = \frac{f_0 + \phi_0 + 2K\left(\frac{1}{y_4^{N+1}} - 1\right)}{2y_3^{\tau}\left[1 - \left(\frac{y_3}{y_4}\right)^{N+1}\right]},$$

$$\delta = \frac{f_0 + \phi_0 + 2K\left(\frac{1}{y_2^{N+1}} - 1\right)}{2y_4^{\tau}\left[1 - \left(\frac{y_4}{y_3}\right)^{N+1}\right]}.$$

Conclusions can now be drawn as to the behavior of $\bar{f}$ and $\bar{\phi}$:

(1) $\bar{f}$ and $\bar{\phi}$ will be real if the constraints are real. This is evident from the fact that y_1^{N+1}, y_2^{N+1}, etc., are real, since $N+1$ is an integer, and,

also the function y^t/y^τ which occurs in every expocyclic member of the solutions is also real, since $t-\tau$ is an integer.

(2) $\bar{f}$ and $\bar{\phi}$ will be cyclical, with discontinuities at the end of each time interval. These discontinuities will arise firstly from the modified reproduction of the constraints in each time interval and secondly from the behavior of the sign of y^t/y^τ. Since $t-\tau$ is an integer, we have $y^t/y^\tau = |y^t/y^\tau| \cos \pi (t-\tau)$, which will be negative where $t-\tau$ is odd and positive where $t-\tau$ is even. This will give rise to successive displacements $\bar{f}$ and $\bar{\phi}$ above and below K. We may adopt the convention that these cycles may be termed of period 2.

(3) Since $y_1{}^t$ and $y_3{}^t$ are explosive, and $y_2{}^t$ and $y_4{}^t$ damped, as $N \to \infty$, α and $\gamma \to 0$, while β and δ remain finite. $N \to \infty$ implies that the competitors are subject to no time restriction in adapting themselves to the situation created by the constraints, and hence is of great economic importance. In this case it is clear that as $t \to \infty$, $y_2{}^t$ and $y_4{}^t$, the only expocyclic or exponential solutions remaining in the solution, will approach zero, so that the limiting value of both $\bar{f}$ and $\bar{\phi}$ is K.

(4) For total output we have

$$\bar{f} + \bar{\phi} = 2(\gamma y_3{}^t + \delta y_4{}^t + K),$$

and, *ceteris paribus*, the amplitude of the cycles will depend on

$$|f_0 + \phi_0 - 2K|.$$

(5) Since $\bar{f} - \phi = 2(\alpha y_1{}^t + \beta y_2{}^t)$, the amplitude of $f \gtrless$ the amplitude of ϕ, according as $\bar{f} \gtrless \bar{\phi}$, assuming f_0 and ϕ_0 do not intersect. And, if $|f_0 - \phi_0|$ is sufficiently great, the cycles will be of different phase.

(6) By considering (14), it can be seen that the smaller n is, the more rapidly will the cycles become damped.

(7) Where $n \to 0$, the value of K approaches the Cournot solution to the static problem. It is also evident that this is the greatest value that K can have.

(8) Where $n \to 1$, the value of K approaches its smallest possible value, namely, the output that would be produced by each competitor if the two acted jointly as a single monopolist.

(9) Where $m = n$, it can be shown that $\bar{f} \equiv \bar{\phi}$ in the range $(0, N)$, while in the range $(-1, 0)$, they are, of course, equal to their respective constraints. In this case any pair of constraints whose sum is identical will yield the same common solution for $\bar{f}$ and $\bar{\phi}$.[12,13]

[12] In all other cases, for the solutions to be equal, the constraints must be identical.

[13] The case where the discriminant of (14) is zero requires treatment of double roots, but as this case has no special economic significance, it will not be considered here.

Our last task is to show that condition (b) is satisfied and that our solution is, in fact, a maximum. It is required to show that

$$\frac{\partial \pi_1}{\partial \omega_1} < 0 \quad \text{for} \quad \omega_1 \neq 0, \omega_2 = 0.$$

Subject to (9), and for $\omega_2 = 0$, (8) becomes

$$\frac{\partial \pi_1}{\partial \omega_1} = \omega_1 \int_0^N [(2a - c)\psi^2 + 2an\psi\psi_{-1}]dt. \tag{23}$$

Now

$$2\psi\psi_{-1} < \psi^2 + \psi_{-1}^2,$$

$$\int_0^N 2\psi\psi_{-1}dt < \int_0^N (\psi^2 + \psi_{-1}^2)dt < \int_0^N 2\psi^2 dt,$$

since for the range (0,1), as we have seen, $\psi_{-1} \equiv 0$. But $2a-c$ and $2an$ are both negative and $|2a-c| > |2an|$. Hence the integrand of (23) is always negative. Similarly

$$\frac{\partial \pi_2}{\partial \omega_2} < 0, \quad \text{for} \quad \omega_2 \neq 0, \omega_1 = 0.$$

III

Case (2). We proceed in exactly the same way as in case (1), now regarding p_1^* and p_2^* as depending on the values of ϕ^* and f^* given by (2a) and (2b). We can then write an equation analogous to (8), namely:

$$\frac{\partial \pi_1}{\partial \omega_1} = \int_0^N [(2a - c)(\bar{f} + \omega_1\psi) + a(\bar{\phi}_{-1} + \omega_2\theta_{-1}) + a(\bar{\phi}_{-1}' + \omega_2\theta_{-1}') + b - d]\psi dt. \tag{24}$$

Hence for condition (a) we have

$$\left(\frac{\partial \pi_1}{\partial \omega_1}\right)_{\omega_1=\omega_2=0} = \int_0^N [(2a - c)\bar{f} + a\bar{\phi}_{-1} + a\bar{\phi}_{-1}' + b - d]\psi dt. \tag{25}$$

Since ψ is arbitrary we must therefore have

$$(2a - c)\bar{f} + a\bar{\phi}_{-1} + a\bar{\phi}_{-1}' + b - d = 0. \tag{26}$$

Proceeding similarly in respect of π_2 we have, analogous to (26),

$$(2a - c)\bar{\phi} + a\bar{f}_{-1} + a\bar{f}_{-1}' + b - d = 0. \tag{27}$$

(26) and (27) are mixed difference and differential equations. Their solution is closely analogous to the solution for negative values of t of

the equation solved by Frisch and Holme[14] and James and Belz.[15] It can be inferred from the argument of James and Belz that, for negative values of t, the infinite sum of characteristic functions which approximates the constraint will not in general converge. Rather than attempting to find particular convergent solutions by James and Belz's method, it is simpler to proceed by the method of successive differentiation.[16]

Let the constraints be denoted by f_0 and ϕ_0 and let $\bar{f}$ and $\bar{\phi}$ at corresponding points in the ranges $(0, 1), (1, 2), \cdots$ be denoted by f_1, ϕ_1; $f_2, \phi_2; \cdots$. Let $a/(2a-c)=\lambda$ and $(b-d)/a=g$. Applying (26) and (27):

$$
\begin{aligned}
f_1 &= -\lambda(f_0 + f_0' + g),\\
\phi_1 &= -\lambda(\phi_0 + \phi_0' + g),\\
f_2 &= \lambda^2(f_0 + 2f_0' + f_0'') + g(-\lambda + \lambda^2),\\
\phi_2 &= \lambda^2(\phi_0 + 2\phi_0' + \phi_0'') + g(-\lambda + \lambda^2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

$$
f_n = (-\lambda)^n(1 + D)^n f_0 + g \sum_{r=1}^{n} (-\lambda)^r, \tag{28}
$$

$$
\phi_n = (-\lambda)^n(1 + D)^n \phi_0 + g \sum_{r=1}^{n} (-\lambda)^r. \tag{29}
$$

These solutions will in general be discontinuous at $t=1, 2, \cdots, n, \cdots$, such that $f(n-0) \gtrless f(n+0)$ and $\phi(n-0) \gtrless \phi(n+0)$, according as n is odd or even, which gives the solution a roughly cyclic character.

Let us first investigate the behavior of f_n and ϕ_n as $n \to \infty$. As $n \to \infty$, the constant term in (26) and (27) will become

$$
-g\frac{\lambda}{1+\lambda} = -\frac{b-d}{3a-c} = K. \tag{30}
$$

Consider next the coefficient of the mth term in the expansion of the binomial term of (28) and (29). This will be in absolute value:

$$
\frac{\lambda^n n!}{(m-n)!m!} < \frac{\lambda^n n^m}{m!}.
$$

Since $0<\lambda<1$, for a given m, $\lambda^n n^m \to 0$ as $n \to \infty$.

Hence we may conclude that, if the constraints have a finite number of nonvanishing derivatives (i.e., they are polynomials), the values of f_n and ϕ_n will approach the constant K, as $n \to \infty$, K as before, being the Cournot solution to the static problem. Further, it must be noted that convergence to the limit K as $n \to \infty$ does not imply that $|K-f|$ is monotonically decreasing throughout the whole range $(0, n)$.

[14] *Op. cit.*
[15] *Op. cit.*
[16] James and Belz, *op. cit.*, p. 327.

From the economic point of view, it would be surprising if a less ambiguous result than the above were achieved, for in many cases it is absurd to suppose that one competitor will assume that his rival will continue to increase output at his present rate of increase. If the constraints are linear, it can easily be seen from (28) and (29) that f and ϕ converge monotonically throughout any range and further a close approximation to linearity in the constraints will ensure fluctuations of decreasing amplitude over a finite period. It may well be argued that these are the only cases where our economic assumptions are justified.

Lastly we must show that condition (b) is satisfied, and that our solutions are, in fact, maxima. Subject to (25) and for $\omega_2=0$, (24) becomes

$$\left(\frac{\partial \pi_1}{\partial \omega_1}\right)_{\omega_2=0} = \omega_1 \int_0^N (2a - c)\psi^2 dt < 0,$$

since $2a - c < 0$. Similarly

$$\left(\frac{\partial \pi_2}{\partial \omega_2}\right)_{\omega_1=0} < 0.$$

CONCLUSION

This paper describes the path by which competitors adapt themselves to new situations. If they are in equilibrium, and then demand or cost conditions change throughout any range ending at $t=0$ and afterwards remain stable, our analysis will describe the process of adaptation to the new conditions; or, alternatively, it will describe the process of adaptation to a change in the methods of estimation of his rival's output by one or both competitors.

The importance of the cycles, to which the problems we have solved give rise, depends, of course, on the length of the lag period. This may be too short for the cycles to be of much significance, at any rate in explaining the business cycle. We prefer to stress the tendency towards equilibrium,[17] which our solutions reveal, as the important point. It seems that the discontinuities are essential. It is reasonable to suppose that there will be a discontinuous break at $t=0$, which is the point at which the confusing period of change ends and the process of adaptation begins. This initial discontinuity will be projected into the future with diminishing force as adaptation continues. The assumption, implicit in our analysis, of a sharp break between change and adaptation is, of course, never exactly fulfilled in actual life.

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[17] Of numerous numerical experiments we have made, it is worth mentioning that, where $n=0$, the Cournot solution is closely approximated after about three cycles, for any figures that appear in any way plausible.

THE APPROXIMATE DETERMINATION OF INDIFFERENCE SURFACES BY MEANS OF ENGEL CURVES*

By A. WALD

LET US BEGIN BY a brief representation of the most important notations. A set of goods $q=(q^1, \cdots, q^n)$ consisting of q^1 units of the good no. 1, $\cdots$, q^n units of the good no. n can be represented by a point q in the n-dimensional Euclidean space with the co-ordinates $q^1, \cdots, q^n$. Each individual has a scale of preferences in this space of goods, that is to say, each individual can say of any two points of the space of goods whether he prefers one to the other, or whether they will be indifferent to him. The scale of preferences can be described by a real function $I(q)$ which associates with every set of goods q a real value $I(q)$ such that $I(q)=I(\bar{q})$ if q and $\bar{q}$ are two indifferent sets of goods and $I(q)>I(\bar{q})$ if q is preferred to $\bar{q}$. Such a function is called an indicator.

Many indicators exist for a given scale of preferences. If $I(q)$ is an indicator then every monotonically increasing function $F(I)$ of I is also an indicator. The set of points q for which the indicator $I(q)$ has a constant value is called an indifference surface. It is obvious that the indifference surfaces do not depend on the choice of the indicator and they are uniquely determined by the scale of preferences.

It is of great practical and theoretical importance to know the shape of the indifference surfaces. If we know the indifference surfaces we are able to solve many problems. We can determine the demand curves for consumers' goods at any price and income, we can calculate the index of cost of living, and so on. The difficulty in determining the indifference surfaces arises from the fact that the indifference surfaces cannot be directly observed, except in the case when we make use of the interview method. But the interview method has many disadvantages and it should be avoided.

We prefer to determine the indifference surfaces by means of observable data, which we have at our disposal. Such observable data are: income, market prices, quantities purchased (consumed). The observable data do not suffice for an exact determination of the indifference surfaces, but approximate determination of them is often possible. The indicator can in many cases be well approximated by a polynomial of the second degree, at least in the small region of the quantity space in

* Paper presented at Colorado Springs, July 12, 1938, before the Cowles Commission Fourth Annual Research Conference on Economics and Statistics. This problem in the special case of two goods has been considered also by Dr. J. Marschak, who presented his results before the Stresa meeting of the Econometric Society in 1934.

which we are interested. In most practical cases we do not wish to know the whole of each indifference surface; it suffices to know the part contained in a certain small region T of the quantity space. Then the assumption is in general justified, that in the region T the indifference surfaces can be well described by a polynomial of the second degree. In exceptional cases where we are interested in knowing the indifference surfaces in a region T, in which the indicator cannot be approximated by a polynomial of the second degree, we shall generally be able to divide the region T in two (or more) regions T_1 and T_2, such that in each region T_1 and T_2 considered separately the approximation by a polynomial of the second degree can be assumed. Therefore we shall make the assumption that the indicator can be approximated by a polynomial of the second degree. Throughout the developments which follow, reference to the region T of the quantity space, in which the above assumption is justified, is always implied.

We understand here by income the total expenditure for consumers' goods. Therefore the income is not an independent datum; it is calculable from prices and quantities purchased. If we denote the quantities purchased by an individual in the time period t by $q^1, \cdots, q^n$ and the prices in the same period by $p^1, \cdots, p^n$, then the income E of this individual in the period t is equal to $\sum p^i q^i$. For a given scale of preferences the quantities purchased are in general uniquely determined by income and prices. If $I(q)$ denotes an indicator, $p^1, \cdots, p^n$ the prices, and E the income, then the quantities purchased, $q^1, \cdots, q^n$, satisfy the equations:

$$(1) \qquad \frac{\partial I}{\partial q^1} \Big/ p^1 = \frac{\partial I}{\partial q^2} \Big/ p^2 = \cdots = \frac{\partial I}{\partial q^n} \Big/ p^n,$$

$$\sum p^i q^i = E.$$

These equations express the fact that among all sets of goods which can be purchased with the amount E, the individual chooses the set for which the indicator $I(q)$ has the greatest value.

Solving the equations (1) we get the quantities purchased as functions of prices and income. If we consider a single time period with constant prices, then the quantities purchased will depend only on the income E, that is to say,

$$(2) \qquad q^1 = f^{(1)}(E), \qquad q^2 = f^{(2)}(E), \qquad \cdots, \qquad q^{(n)} = f^{(n)}(E).$$

These equations represent a curve in the n-dimensional quantity space, which is called an Engel curve. To each system of prices belongs a certain Engel curve.

The Engel curves can be determined empirically in each time period

by observing the consumption of individuals belonging to different income classes. The determination of the Engel curves in this way is correct only if we can suppose that the individuals considered have the same scale of preferences. This condition, of course, can only be fulfilled approximately.

I. THE CONDITIONS OF INTEGRABILITY AND THE CALCULATION OF THE INDICATOR $I(q)$

Let us consider the Engel curves $C_1, \cdots, C_k$ belonging to the time periods $t_1, \cdots, t_k$, or, more generally, to the different price situations nos. $1, 2, \cdots, k$. Denote the market prices of the various goods in the time period t_i $(i=1, 2, \cdots, k)$ by $\bar{p}_i^1, \bar{p}_i^2, \cdots, \bar{p}_i^n$. On account of our assumption that the indifference surfaces can be described by an indicator of the second degree, the Engel curves must be linear and each Engel curve C_i $(i=1, \cdots, k)$ is therefore determined by two of its points, say

$$\bar{q}_i = (\bar{q}_i^1, \cdots, \bar{q}_i^n) \quad \text{and} \quad \bar{\bar{q}}_i = (\bar{\bar{q}}_i^1, \cdots, \bar{\bar{q}}_i^n)), \qquad (i = 1, \cdots, k).$$

For the sake of simplification, it is advantageous to introduce the following notation: Denote $\bar{q}_i$ by q_{2i-2} and $\bar{\bar{q}}_i$ by q_{2i-1}. For the system of prices $\bar{p}_i = (\bar{p}_i^1, \cdots, \bar{p}_i^n)$ we shall simultaneously use the symbols p_{2i-2} and p_{2i-1}. There can be no confusion in using two different notations for the same system of prices $\bar{p}_i$. In fact if i and j are two different integers, then we have for $\bar{p}_i$ the symbols p_{2i-2} and p_{2i-1}, and for $\bar{p}_j$ the symbols p_{2j-2} and p_{2j-1}. Since $i \neq j$ the following inequalities,

$$2i - 2 \neq 2j - 2,\ 2i - 2 \neq 2j - 1,\ 2i - 1 \neq 2j - 2,\ 2i - 1 \neq 2j - 1,$$

hold true. Hence $\bar{p}_i$ and $\bar{p}_j$ will always have different notations.

Let us consider the vector $v_i = q_0 q_i$ with the initial point q_0 and terminal point q_i $(i=1, 2, \cdots, 2k-1)$. The index i of q_i takes the values $1, 2, \cdots, 2k-1$, because, for $i=1, \cdots, k$, we denoted $\bar{q}_i$ by q_{2i-2} and $\bar{\bar{q}}_i$ by q_{2i-1}. We shall prove the following:

THEOREM I. *If there exists an indicator $I(q)$ which is a polynomial of the second degree in q over the $(2k-1)$-dimensional linear space determined by the vectors $v_1, \cdots, v_{2k-1}$ associated with the Engel curves $C_1, \cdots, C_k$, then this indicator is uniquely determined*[1] *in S by the said Engel curves, apart from an arbitrary proportionality factor and an arbitrary additive constant.*

Let us denote the components of the vector v_i by $v_i^1, v_i^2, \cdots, v_i^n$.

[1] In the developments which follow, we shall use the expression "uniquely determined" also for functions which are determined with the exception of a proportionality factor and an additive constant.

Hence $v_i^j = q_i^j - q_0^j$ $(j=1, 2, \cdots, n)$. For each point q of the linear space S around q_0 determined by the vectors $v_1, \cdots, v_{2k-1}$, the vector $v = q_0q$ with the initial point q_0 and terminal point q can be represented as a vector sum $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{2k-1} v_{2k-1}$, where $\lambda_1, \cdots, \lambda_{2k-1}$ are real numbers. The numbers $\lambda_1, \cdots, \lambda_{2k-1}$ we call the vector co-ordinates of the point q. If the vectors $v_1, \cdots, v_{2k-1}$ are linearly independent, then the vector co-ordinates of a point q of the linear space S are uniquely determined. If the vectors $v_1, \cdots, v_{2k-1}$ are linearly dependent, then for each given point q of S there exist an infinite number of systems of values $\lambda_1, \cdots, \lambda_{2k-1}$ such that $\lambda_1 v_1 + \cdots + \lambda_{2k-1} v_{2k-1} = v = q_0q$. But to each given system $\lambda_1, \cdots, \lambda_{2k-1}$ of vector co-ordinates belongs in any case only a single point q, namely the terminal point of the vector $\lambda_1 v_1 + \cdots + \lambda_{2k-1} v_{2k-1}$.

Denote the marginal utility of money at the point q_i under the system of prices p_i by w_i $(i=0, 1, \cdots, 2k-1)$. Hence

$$w_i = \frac{\partial I(q_i)}{\partial q_i^1} \Big/ p_i^1 = \cdots = \frac{\partial I(q_i)}{\partial q_i^n} \Big/ p_i^n.$$

The indicator $I(q)$ can be represented in S by a function $f(\lambda_1, \cdots, \lambda_{2k-1})$ of the vector co-ordinates $\lambda_1, \cdots, \lambda_{2k-1}$, that is, $f(\lambda_1, \cdots, \lambda_{2k-1})$ is equal to $I(q)$, where q is the terminal point of the vector $\lambda_1 v_1 + \cdots + \lambda_{2k-1} v_{2k-1}$.

The derivatives of the indicator $I(q)$ at the points $q_0, q_1, \cdots, q_{2k-1}$ in the direction of the vector v_i are given by the following expressions:

$$(3)\quad \begin{cases} \dfrac{\partial I(q_0)}{\partial \lambda_i} = \dfrac{\partial f(0, 0, \cdots, 0)}{\partial \lambda_i} = w_0(p_0^1 v_i^1 + \cdots + p_0^n v_i^n), \\ \dfrac{\partial I(q_1)}{\partial \lambda_i} = \dfrac{\partial f(1, 0, 0, \cdots, 0)}{\partial \lambda_i} = w_1(p_1^1 v_i^1 + \cdots + p_1^n v_i^n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial I(q_{2k-1})}{\partial \lambda_i} = \dfrac{\partial f(0, 0, \cdots, 0, 1)}{\partial \lambda_i} = w_{2k-1}(p_{2k-1}^1 v_i^1 + \cdots + p_{2k-1}^n v_i^n). \end{cases}$$

The function $f(\lambda_1, \cdots, \lambda_{2k-1})$ is also a polynomial of the second degree, like $I(q)$, and therefore the partial derivatives of $f(\lambda_1, \cdots, \lambda_{2k-1})$ with respect to $\lambda_1, \cdots, \lambda_{2k-1}$ are linear functions of $\lambda_1, \cdots, \lambda_{2k-1}$. Hence we get easily from (3).

$$(4)\quad \begin{aligned} \frac{\partial f(\lambda_1, \cdots, \lambda_{2k-1})}{\partial \lambda_i} &= w_0 \sum_{r=1}^{n} p_0^r v_i^r + \lambda_1 \sum_{r=1}^{n} (w_1 p_1^r v_i^r - w_0 p_0^r v_i^r) + \cdots \\ &\quad + \lambda_{2k-1} \sum_{r=1}^{n} (w_{2k-1} p_{2k-1}^r v_i^r - w_0 p_0^r v_i^r). \end{aligned}$$

If we denote $\sum_{r=1}^{n} p^r \partial v_j^r$ by $\rho_{i,j}$, we get

$$\frac{\partial f(\lambda_1, \cdots, \lambda_{2k-1})}{\partial \lambda_i} = w_0\rho_{0,i} + \lambda_1(w_1\rho_{1,i} - w_0\rho_{0,i}) + \cdots + \lambda_{2k-1}(w_{2k-1}\rho_{2k-1,i} - w_0\rho_{0,i}). \tag{5}$$

Hence

$$\frac{\partial^2 f(\lambda_1, \cdots, \lambda_{2k-1})}{\partial \lambda_i \partial \lambda_j} = w_j\rho_{j,i} - w_0\rho_{0,i}, \quad \begin{matrix}(i = 1, \cdots, 2k-1),\\ (j = 1, \cdots, 2k-1).\end{matrix} \tag{6}$$

Since

$$\frac{\partial^2 f}{\partial \lambda_i \lambda_j} = \frac{\partial^2 f}{\partial \lambda_j \partial \lambda_i},$$

we get the equations

$$w_i\rho_{i,j} - w_0\rho_{0,j} = w_j\rho_{j,i} - w_0\rho_{0,i}, \quad \begin{matrix}(i = 1, \cdots, 2k-1),\\ (j = 1, \cdots, 2k-1).\end{matrix} \tag{7}$$

The value of $\rho_{i,j}$ is empirically known for each i and j. It can be calculated by means of the market prices in the time period t_i and of the quantities consumed in the periods t_j and t_0. But the marginal utilities of money, $w_0, w_1, \cdots, w_{2k-1}$, are unknown. We have to determine them by means of the equations (7). The equations (7) are homogeneous in the unknown quantities $w_0, w_1, \cdots, w_{2k-1}$. If $w_1', \cdots, w_{2k-1}'$ is a solution, then for any real value μ, the system of values $\mu w_0', \mu w_1', \cdots, \mu w_{2k-1}$ is also a solution. Hence we can choose one of the quantities $w_0, w_1, \cdots, w_{2k-1}$ arbitrarily. Let w_0 be equal to 1. By choosing w_0 equal to 1, we have merely fixed the value of the proportionality factor of the indicator $I(q)$. For the determination of the $2k-1$ unknown quantities $w_1, \cdots, w_{2k-1}$ we have at our disposal $\binom{2k-1}{2}$ (combinations of $2k-1$ things taken 2 at a time) equations. The case $k=1$ is not interesting; we shall consider only the case $k \geqq 2$. If $k>2$ we have more equations than unknown quantities. If our assumption that the indicator is a polynomial of the second degree were exactly true and if our observations were without error, then it would be possible to choose the quantities $w_1, \cdots, w_{2k-1}$ such that the equations (7) are exactly fulfilled. But this is certainly not the case. There are many sources of errors. Not all individuals considered have exactly the same scale of preferences, the individuals do not choose exactly the set of goods for which the indicator has the greatest value, the goods compared in different periods are not exactly of the same quality, and so on. Different errors will arise even in the statistical determination

of prices and quantities. For all these reasons we cannot expect that the equations (7) can be fulfilled exactly. Let us denote $w_i\rho_{i,j}-w_0\rho_{0,j}$ by $\alpha_{i,j}$. Then the equations (7) can be written as follows:

$$\alpha_{i,j} = \alpha_{j,i}, \tag{7*}$$

that is to say, the determinant

$$\Delta = \begin{vmatrix} \alpha_{1,1}, & \alpha_{2,1}, & \cdots, & \alpha_{2k-1,1} \\ \alpha_{1,2}, & \alpha_{2,2}, & \cdots, & \alpha_{2k-1,2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1,2k-1}, & \alpha_{2,2k-1}, & \cdots, & \alpha_{2k-1,2k-1} \end{vmatrix}$$

must be symmetrical. Since we cannot expect that the determinant can be made exactly symmetrical by adequate choice of $w_1, \cdots, w_{2k-1}$, we shall choose $w_1, \cdots, w_{2k-1}$ such that the deviation from symmetry becomes as small as possible. We can introduce as a measure of asymmetry of a determinant a_{ij} the expression

$$\delta = \sum_i \sum_j (a_{ij} - a_{ji})^2 \quad \text{or} \quad \delta = \sum_i \sum_j \left(\frac{a_{ij} - a_{ji}}{|a_{ij}| + |a_{ji}|} \right)^2.$$

We have to calculate the values $w_1, \cdots, w_{2k-1}$ which make δ a minimum. But this calculation is in general too complicated. For practical purposes it will be sufficient to calculate $w_1, \cdots, w_{2k-1}$ from the first $2k-1$ equations of (7), namely from the equations:

$$\begin{aligned} w_2\rho_{2,1} - \rho_{0,1} &= w_1\rho_{1,2} - \rho_{0,2}, \\ w_3\rho_{3,1} - \rho_{0,1} &= w_1\rho_{1,3} - \rho_{0,3}, \\ &\cdots \\ w_{2k-1}\rho_{2k-1,1} - \rho_{0,1} &= w_1\rho_{1,2k-1} - \rho_{0,2k-1}, \\ w_3\rho_{3,2} - \rho_{0,2} &= w_2\rho_{2,3} - \rho_{0,3}. \end{aligned} \tag{8}$$

If our assumptions are approximately true then the values $w_1, \cdots, w_{2k-1}$ thus calculated should come near to the values which make δ a minimum.

Solving the first $2k-2$ equations of (8) we get

$$w_i = \frac{w_1\rho_{1,i} - \rho_{0,i} + \rho_{0,1}}{\rho_{i,1}}, \qquad (i = 2, \cdots, 2k-1). \tag{9}$$

Substituting in the last equation of (8) for w_2 and w_3 their values from (9), we get

$$(10)\qquad w_1 = \frac{\rho_{0,2} - \rho_{0,3} + \rho_{2,3}\dfrac{\rho_{0,1}-\rho_{0,2}}{\rho_{2,1}} - \rho_{3,2}\dfrac{\rho_{0,1}-\rho_{0,3}}{\rho_{3,1}}}{\rho_{3,2}\dfrac{\rho_{1,3}}{\rho_{3,1}} - \rho_{2,3}\dfrac{\rho_{1,2}}{\rho_{2,1}}}.$$

The denominator on the right-hand side of (9) is equal to $\rho_{i,1} = \sum p_i^r(q_1^r - q_0^r)$. Since q_1 and q_0 lie on the same Engel curve, the expression $\rho_{i,1}$ will deviate considerably from zero, provided that we choose q_1 not very near to q_0. Hence the equation (9) can be well applied to the calculation of the marginal utilities of money, $w_2, \cdots, w_{2k-1}$, as functions of w_1. Small errors in $\rho_{i,j}$ will not greatly change the values of w_i $(i=2, \cdots, 2k-1)$. But we cannot say the same about the equation (10). The denominator on the right-hand side of (10) will often be near to zero and we cannot determine the value of w_1 with great accuracy. Small changes in the values of $\rho_{i,j}$ will perhaps change greatly the value of w_1. The reason for this fact is the following: We know that every monotonically increasing function $F(I)$ of I is also an indicator. We get here a unique solution for the indicator because we consider that indicator which is a polynomial of the second degree. It is true that $F(I)$ is a second-degee polynomial only if F is linear, but $F(I)$ can be well approximated in a small region by a polynomial of the second degree, even if $F(I)$ is not linear, for instance, if $F(I)=\log I$ or $=(I)^2$. The polynomial of the second degree I^* which approximates $F(I)$, taken as an indicator, will describe a system of indifference surfaces which will not be identical with that described by I, but they will not differ much. However, the value of w_1 can be quite different if we take I^* instead of I as an indicator. The circumstance that w_1 cannot be determined with good accuracy does not mean any disadvantage. A change in the value of w_1 means only that we get I^* instead of I as an indicator, where I^* is a polynomial of the second degree, which approximates a certain function $F(I)$ of I. The indifference surfaces themselves will be very little affected by it.

The measure of asymmetry δ can be calculated after having determined $w_1, \cdots, w_{2k-1}$. If we make assumptions about the nature of the errors, we can deduce statistical criteria when δ can be regarded as significant. This question will not be discussed here. If δ is not to be regarded as significant, we substitute for the matrix $\triangle$ a symmetrical one, substituting for the elements $\alpha_{i,j}=w_i\rho_{i,j}-\rho_{0,j}$ and $\alpha_{j,i}=w_j\rho_{j,i}-\rho_{0,i}$ their arithmetic mean $\bar{\alpha}_{i,j}=\frac{1}{2}(\alpha_{i,j}+\alpha_{j,i})$.

Since according to (6)

$$\frac{\partial^2 f}{\partial\lambda_i\partial\lambda_j} = \alpha_{j,i} \quad \text{and} \quad \frac{\partial^2 f}{\partial\lambda_j\partial\lambda_i} = \alpha_{i,j},$$

the value $\bar{a}_{i,j}$ is taken as the corrected common value of

$$\frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \quad \text{and} \quad \frac{\partial^2 f}{\partial \lambda_j \partial \lambda_i}.$$

Since according to (5) the constant term in $\partial f/\partial \lambda_i$ is equal to $\rho_{0,i}$, we get the following formula for the indicator

$$(11) \quad f(\lambda_1, \cdots, \lambda_{2k-1}) = \frac{1}{2} \sum_{i=1}^{2k-1} \sum_{j=1}^{2k-1} \bar{a}_{i,j} \lambda_i \lambda_j + \sum_{i=1}^{2k-1} \rho_{0,i} \lambda_i + C,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_{2k-1}$ are the vector co-ordinates of the point considered, and C is a constant, which we can choose arbitrarily.

The equation (11) gives us the indicator as a function of the vector co-ordinates $\lambda_1, \cdots, \lambda_{2k-1}$. Practically we want to have the indicator expressed as a function $I(q^1, \cdots, q^n)$ of the quantities $q^1, \cdots, q^n$, that is to say, as a function of the Cartesian co-ordinates of the point q. The transformation of $f(\lambda_1, \cdots, \lambda_{2k-1})$ into the form $I(q^1, \cdots, q^n)$ can be made as follows:

We consider the equations

$$(12) \quad \begin{aligned} \lambda_1 v_1^1 + \lambda_2 v_2^1 + \cdots + \lambda_{2k-1} v_{2k-1}^1 &= q^1 - q_0^1, \\ \lambda_1 v_1^2 + \lambda_2 v_2^2 + \cdots + \lambda_{2k-1} v_{2k-1}^2 &= q^2 - q_0^2, \\ \cdots\cdots\cdots\

vectors $v_{b_1}, \cdots, b_{b_r}$. Hence the linear space S determined by the vectors $v_1, \cdots, v_{2k-1}$ is identical with the linear space determined by the vectors $v_{b_1}, \cdots, v_{b_r}$. Therefore we can put $\lambda_i = 0$ for every i which is not equal to any of the numbers $b_1, \cdots, b_r$, and we have to consider only the r equations from (12):

$$
\begin{aligned}
&\lambda_{b_1} v_{b_1}^{a_1} + \lambda_{b_2} v_{b_2}^{a_1} + \cdots + \lambda_{b_r} v_{b_r}^{a_1} = q^{a_1} - q_0^{a_1}, \\
&\lambda_{b_1} v_{b_1}^{a_2} + \lambda_{b_2} v_{b_2}^{a_2} + \cdots + \lambda_{b_r} v_{b_r}^{a_2} = q^{a_2} - q_0^{a_2}, \\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\lambda_{b_1} v_{b_1}^{a_r} + \lambda_{b_2} v_{b_2}^{a_r} + \cdots + \lambda_{b_r} v_{b_r}^{a_r} = q^{a_r} - q_0^{a_r}.
\end{aligned} \tag{13}
$$

The determinant of these equations by hypothesis is not zero. Hence we get $\lambda_{b_1}, \cdots, \lambda_{b_r}$ as functions of $q^{a_1}, \cdots, q^{a_r}$, that is to say,

$$
\begin{cases}
\lambda_{b_1} = f_1(q^{a_1}, \cdots, q^{a_r}), \\
\lambda_{b_2} = f_2(q^{a_1}, \cdots, q^{a_r}), \\
\qquad \cdots\cdots\cdots \\
\lambda_{b_r} = f_r(q^{a_1}, \cdots, q^{a_r}).
\end{cases} \tag{14}
$$

Substituting 0 for λ in the right-hand side of (11), if i is not equal to any of the numbers $b_1, \cdots, b_r$, and the expression $f_j(q^{a_1}, \cdots, q^{a_r})$ for λ_{b_j}, we get the indicator as a function of the quantities $q^{a_1}, \cdots, q^{a_r}$, that is to say,

$$I(q) = \phi(q^{a_1}, \cdots, q^{a_r}).$$

It may look strange at first sight that the indicator is represented as a function of the quantities of the goods $G^{a_1}, \cdots, G^{a_r}$ and seems to be independent of the quantities of the other goods. But if we keep in mind that this representation of the indicator holds true *only in the linear space S determined by the vectors* $v_{b_1}, \cdots, v_{b_r}$, we see immediately that $I(q)$ is in reality also a function of the other goods. In fact, a set of goods $q = (q^1, \cdots, q^n)$ which lies in S is uniquely determined by its co-ordinates $q^{a_1}, \cdots, q^{a_r}$. If q lies in S, then for each integer $i \leqq n$ and $\neq a_1, \cdots, a_r$, q^i is a certain linear function of $q^{a_1}, \cdots, q^{a_r}$ which can be calculated from the equations (12). Let us denote this function by $\psi_i(q^{a_1}, \cdots, q^{a_r})$ $(i \neq a_1, \cdots, a_r)$. If $I(q^1, \cdots, q^n)$ is the indicator in the whole quantity space, we get our indicator $I(q^{a_1}, \cdots, q^{a_r})$ by substituting in $I(q^1, \cdots, q^n)$ for q^i $(i \neq a_1, \cdots, a_r)$ the expression $\psi_i(q^{a_1}, \cdots, q^{a_r})$.

If the rank of the matrix M is equal to n, then the set of integers $a_1, a_2, \cdots, a_r$ is equal to the set $1, 2, \cdots, n$ and we get the indicator in the whole quantity space expressed as a function of $q^1, \cdots, q^n$.

If we substitute in (7) for w_i $(i=2, 3, \cdots, 2k-1)$ the right-hand side of (9) we get

$$\rho_{i,j}\frac{w_1\rho_{1,i}-\rho_{0,i}+\rho_{0,1}}{\rho_{i,1}}-\rho_{0,j}=\rho_{j,i}\frac{w_1\rho_{1,j}-\rho_{0,j}+\rho_{0,1}}{\rho_{j,1}}-\rho_{0,i}, \tag{15}$$

$$(i=2, \cdots, 2k-1), (j=2, \cdots, 2k-1).$$

In these equations w_1 is the single unknown quantity. If we substitute in (15) for w_1, its expression from (10), we get relations between the values $\rho_{i,j}$. We shall call the relations (15) *conditions of integrability.*

In case $k=2$, that is to say, if we consider only 2 time periods t_1 and t_2 and the corresponding two Engel curves C_1 and C_2, then we have no conditions of integrability. In fact, the equations (15) will be reduced in this case to a single equation, namely to the equation in which $i=2$ and $j=3$ (the equation in which $i=3$ and $j=2$ is identical with this). This single equation suffices only to determine the value w_1 and therefore we have no conditions of integrability.

Let us illustrate the calculation of the indicator by a numerical example. We consider 3 consumers' goods, nos. 1, 2, and 3, and two time periods, t_1, t_2. The prices in the two time periods are given as follows:

$$\bar{p}_1^1=1,\ \bar{p}_1^2=1,\ \bar{p}_1^3=1,\ \bar{p}_2^1=\tfrac{1}{2},\ \bar{p}_2^2=1,\ \bar{p}_2^3=3.$$

The Engel curve C_1 is given by the equations:

$$q^1=\tfrac{1}{3}E_1,\ q^2=\tfrac{1}{3}E_1,\ q^3=\tfrac{1}{3}E_1;$$

and the Engel curve C_2 by the equations:

$$q^1=\tfrac{1}{3}E_2+1,\ q^2=\tfrac{1}{3}E_2-\tfrac{1}{2},\ q^3=\tfrac{1}{6}E_2,$$

where E_1 denotes the income in the period 1 and E_2 the income in the period 2.

We have to choose on each Engel curve two points. We choose on C_1 the points q_0 and q_1 corresponding to the incomes 0 and 3, and on C_2 the points q_2 and q_3 corresponding to the incomes 3 and 6. Hence we have:

$$q_0=(0, 0, 0), \qquad q_1=(1, 1, 1),$$
$$q_2=(2, 0.5, 0.5), \qquad q_3=(3, 1.5, 1),$$
$$p_0=\bar{p}_1=(1, 1, 1), \qquad p_1=\bar{p}_1=(1, 1, 1),$$
$$p_2=\bar{p}_2=(0.5, 1, 3), \qquad p_3=\bar{p}_2=(0.5, 1, 3).$$

Now we have to calculate the values $\rho_{i,j}$. Since $\rho_{i,j}$ is equal to $\sum_{r=1}^{n}p_i^r q_j^r$ $=\sum_{r=1}^{n}p_i^r(q_j^r-q_0^r)$, we get

$$(16)\qquad \begin{aligned} &\rho_{0,1} = 3, && \rho_{0,2} = 3, && \rho_{0,3} = 5.5,\\ &\rho_{1,1} = 3, && \rho_{1,2} = 3, && \rho_{1,3} = 5.5,\\ &\rho_{2,1} = 4.5, && \rho_{2,2} = 3, && \rho_{2,3} = 6,\\ &\rho_{3,1} = 4.5, && \rho_{3,2} = 3, && \rho_{3,3} = 6. \end{aligned}$$

Substituting in (10) for $\rho_{i,j}$ the numerical values from above, we get

$$w_1 = \frac{3 - 5.5 + 6\dfrac{3-3}{4.5} - 3\dfrac{3-5.5}{4.5}}{3\dfrac{5.5}{4.5} - 6\dfrac{3}{4.5}} = 2.5.$$

Hence from (9) we get

$$w_2 = \frac{(2.5)3 - 3 + 3}{4.5} = \frac{5}{3}$$

and

$$w_3 = \frac{(2.5)5.5 - 5.5 + 3}{4.5} = 2.5.$$

Since we have considered only two time periods, we have no conditions of integrability and, if we substitute for w_1, w_2, w_3 their calculated numerical values, $\alpha_{i,j}$ becomes equal to $\alpha_{j,i}$, where $\alpha_{ij} = w_i\rho_{i,j} - \rho_{0,j}$.

In order to determine the indicator we have first to calculate the values $\alpha_{i,j}$.

$$(17)\qquad \begin{cases} \alpha_{1,1} = w_1\rho_{1,1} - \rho_{0,1} = (2.5)3 - 3 = 4.5,\\ \alpha_{1,2} = w_1\rho_{1,2} - \rho_{0,2} = (2.5)3 - 3 = 4.5,\\ \alpha_{1,3} = w_1\rho_{1,3} - \rho_{0,3} = (2.5)5.5 - 5.5 = 8.25,\\ \alpha_{2,2} = w_2\rho_{2,2} - \rho_{0,2} = (\tfrac{5}{3})3 - 3 = 2,\\ \alpha_{2,3} = w_2\rho_{2,3} - \rho_{0,3} = (\tfrac{5}{3})6 - 5.5 = 4.5,\\ \alpha_{3,3} = w_3\rho_{3,3} - \rho_{0,3} = (2.5)6 - 5.5 = 9.5. \end{cases}$$

According to (11) we get

$$(18)\qquad \begin{aligned} f(\lambda_1, \lambda_2, \lambda_3) &= \tfrac{1}{2}\sum\sum \alpha_{i,j}\lambda_i\lambda_j + \sum \rho_{0,i}\lambda_i\\ &= 2.25\lambda_1^2 + \lambda_2^2 + 4.75\lambda_3^2 + 4.5\lambda_1\lambda_2 + 8.25\lambda_1\lambda_3\\ &\quad + 4.5\lambda_2\lambda_3 + 3\lambda_1 + 3\lambda_2 + 5.5\lambda_3, \end{aligned}$$

where $f(\lambda_1, \lambda_2, \lambda_3)$ is the indicator and λ_1, λ_2, λ_3 are the vector co-ordinates with respect to the system of vectors $v_1 = q_0q_1$, $v_2 = q_0q_2$, and

$v_3 = q_0 q_3$. If we want to express the indicator as a function of q^1, q^2, and q^3, we have to solve the equations (12). Since the initial point of the vector v_i $(i=1, 2, 3)$ is q_0 and the terminal point q_i $(i=1, 2, 3)$, we get

$$\begin{aligned} \overset{1}{v}_1 &= 1, & \overset{2}{v}_1 &= 1, & \overset{3}{v}_1 &= 1, \\ \overset{1}{v}_2 &= 2, & \overset{2}{v}_2 &= \tfrac{1}{2}, & \overset{3}{v}_2 &= \tfrac{1}{2}, \\ \overset{1}{v}_3 &= 3, & \overset{2}{v}_3 &= 1.5, & \overset{3}{v}_3 &= 1, \end{aligned}$$

and the equations (12) become

$$\begin{aligned} \lambda_1 + \ 2\lambda_2 + 3\lambda_3 &= \overset{1}{q} - \overset{1}{q}_0 = \overset{1}{q}, \\ \lambda_1 + 0.5\lambda_2 + 1.5\lambda_3 &= \overset{2}{q} - \overset{2}{q}_0 = \overset{2}{q}, \\ \lambda_1 + 0.5\lambda_2 + \lambda_3 &= \overset{3}{q} - \overset{3}{q}_0 = \overset{3}{q}. \end{aligned}$$

Since the determinant of these equations is not equal to zero, the values λ_1, λ_2, λ_3 can be calculated. We get:

$$\begin{aligned} \lambda_1 &= -\tfrac{1}{3}q^1 - \tfrac{2}{3}q^2 + 2q^3, \\ \lambda_2 &= \tfrac{2}{3}q^1 - \tfrac{8}{3}q^2 + 2q^3, \\ \lambda_3 &= 2q^2 - 2q^3. \end{aligned}$$

If we substitute in (18) for λ_1, λ_2, λ_3 the right-hand sides of the above equations, we get the indicator as a function of q^1, q^2, q^3.

Remark: The fulfilment of the integrability conditions (15) means that there exists one and only one indicator $I(q)$ (with exception of a proportionality factor and an additive constant) such that along the given Engel curves $C_1, \cdots, C_k$ the necessary conditions for equilibrium position are fulfilled, that is to say that the equations

$$\frac{\partial I}{\partial q^1} \Big/ \overset{1}{p}_i = \frac{\partial I}{\partial q^2} \Big/ \overset{2}{p}_i = \cdots = \frac{\partial I}{\partial q^n} \Big/ \overset{n}{p}_i, \qquad (i = 1, 2, \cdots, k),$$

hold true. But the fulfilment of the integrability conditions (15) does not imply the fulfilment of the sufficient condition of equilibrium, namely that the second differential of $I(q)$, subject to $\sum p^r q^r = E$, is a negative definite quadratic form. It can happen that although the conditions (15) are fulfilled, we get an indicator of which the second differerential, subject to $\sum p^r q^r = E$, is an indefinite quadratic form and therefore the given Engel curves cannot be described by an indicator, just as if the integrability conditions (15) were not fulfilled. Therefore it is important to know whether for the indicator given by formula (11) the sufficient conditions of equilibrium along the Engel curves are fulfilled.

In order to see whether the sufficient conditions of equilibrium along the Engel curves C_i are satisfied, we shall calculate the second differential of $I(q)$, subject to $\sum p_i^r q^r = E_i$, where p_i denotes the system of prices in the period i. It is more convenient to express the second differential in terms of the vector co-ordinates $\lambda_1, \cdots, \lambda_{2k-1}$. Therefore we have to express the condition $\sum p_i^r q^r = E_i$ in terms of the vector co-ordinates $\lambda_1, \cdots, \lambda_{2k-1}$. It is obvious that for each point $q = (q^1, \cdots, q^n)$ of the linear space S determined by the vectors $v_1, \cdots, v_{2k-1}$ the equation

$$\sum \overset{r}{p}_i \overset{r}{q} = \sum \overset{r}{p}_i \overset{r}{q}_0 + \rho_{i,1}\lambda_1 + \cdots + \rho_{i,2k-1}\lambda_{2k-1}$$

holds true, where $\lambda_1, \cdots, \lambda_{2k-1}$ denote the vector co-ordinates of $q = (q^1, \cdots, q^n)$.

Hence we have to consider the condition

$$\sum_{\nu=1}^{2k-1} \rho_{i,\nu}\lambda_\nu = \text{constant}. \tag{19}$$

If $\rho_{i,2k-1} \neq 0$, we can write (19) as follows:

$$\lambda_{2k-1} = \text{constant} - \frac{\rho_{i,1}\lambda_1 + \cdots + \rho_{i,2k-2}\lambda_{2k-2}}{\rho_{i,2k-1}}. \tag{20}$$

If $f(\lambda_1, \cdots, \lambda_{2k-1})$ denotes the indicator and $f^{(i)}$ $(\lambda_1, \cdots, \lambda_{2k-2})$ the function which we get by substituting in $f(\lambda_1, \cdots, \lambda_{2k-1})$ for λ_{2k-1}, the expression on the right-hand side of (20), then it can be easily shown that

$$\frac{\partial^2 f^{(i)}}{\partial\lambda_\mu \partial\lambda_\nu} = \frac{\partial^2 f}{\partial\lambda_\mu \partial\lambda_\nu} - \frac{\partial^2 f}{\partial\lambda_\nu \partial\lambda_{2k-1}} \frac{\rho_{i,\mu}}{\rho_{i,2k-1}} - \frac{\partial^2 f}{\partial\lambda_\mu \partial\lambda_{2k-1}} \frac{\rho_{i,\nu}}{\rho_{i,2k-1}} + \frac{\partial^2 f}{\partial\lambda_{2k-1}^2} \cdot \frac{\rho_{i,\mu}\rho_{i,\nu}}{(\rho_{i,2k-1})^2}. \tag{21}$$

According to (6) and (7)[2]

$$\frac{\partial^2 f}{\partial\lambda_\mu \partial\lambda_\nu} = \alpha_{\nu,\mu} = \alpha_{\mu,\nu} = w_\mu \rho_{\mu,\nu} - \rho_{0,\nu}.$$

Hence we can write (21) in the following form:

[2] We suppose here that the integrability conditions (15) are fulfilled. If (15) does not hold true, then even the necessary conditions of equilibrium are not fulfilled and therefore the investigation of the second differential has no sense.

$$\frac{\partial^2 f^{(i)}}{\partial\lambda_\mu \partial\lambda_\nu} = \overset{(i)}{\alpha}_{\mu,\nu} = \alpha_{\mu\nu} - \frac{\rho_{i,\mu}}{\rho_{i,2k-1}}\alpha_{\nu,2k-1} - \frac{\rho_{i,\nu}}{\rho_{i,2k-1}}\alpha_{\mu,2k-1} + \frac{\rho_{i,\mu}\rho_{i,\nu}}{(\rho_{i,2k-1})^2}\alpha_{2k-1,2k-1}. \tag{22}$$

Hence the second differential of $f(\lambda_1, \cdots, \lambda_{2k-1})$, subject to (20), is the quadratic form

$$Q^{(i)} = \sum_{\mu=1}^{2k-2}\sum_{\nu=1}^{2k-2} \overset{(i)}{\alpha}_{\mu,\nu} d\lambda_\mu d\lambda_\nu. \tag{23}$$

Since the indicator $f(\lambda_1, \cdots, \lambda_{2k-1})$ is a polynomial of the second degree, the necessary and sufficient conditions that $I(q)$, subject to $\sum p_i^r q^r = E_i$, has a maximum value at the point $q = q^1, \cdots, q^n$ can be formulated as follows:

(a) $\dfrac{\partial I}{\partial q^1}\Big/ \overset{1}{p}_i = \dfrac{\partial I}{\partial q^2}\Big/ \overset{2}{p}_i = \cdots = \dfrac{\partial I}{\partial q^n}\Big/ \overset{n}{p}_i.$

(b) $\sum \overset{r}{p}_i \overset{r}{q} = E_i.$

(c) $Q^{(i)}$ is a negative definite or semi-definite quadratic form.

If $I(q)$ should not be a polynomial of the second degree, then only the condition that $Q^{(i)}$ [together with (a) and (b)] is negative definite would be sufficient, but not the condition that $Q^{(i)}$ is negative semi-definite.

The quadratic form $Q^{(i)}$ is negative definite (or semidefinite) only in the case that in the sequence

$$\left|\overset{(i)}{\alpha}_{1,1}\right|, \begin{vmatrix} \overset{(i)}{\alpha}_{1,1}, & \overset{(i)}{\alpha}_{1,2} \\ \overset{(i)}{\alpha}_{2,1}, & \overset{(i)}{\alpha}_{2,2} \end{vmatrix}, \cdots, \begin{vmatrix} \overset{(i)}{\alpha}_{1,1}, & \cdots, & \overset{(i)}{\alpha}_{1,2k-2} \\ \overset{(i)}{\alpha}_{2,1}, & \cdots, & \overset{(i)}{\alpha}_{2,2k-2} \\ \cdots & \cdots & \cdots \\ \overset{(i)}{\alpha}_{2k-2,1}, & \cdots, & \overset{(i)}{\alpha}_{2k-2,2k-2} \end{vmatrix},$$

of the principal minors of $\left|\alpha^{(i)}_{\mu,\nu}\right|$, all minors of even order are $\geqq 0$ and all minors of odd order $\leqq$ zero.

We can summarize our results in the following

THEOREM II: *We consider k time periods, $t_1, \cdots, t_k$, in each time period $t_i (i = 1, \cdots, k)$ being given the system of prices, $p_i = (p_i^1, \cdots, p_i^n)$, and a straight line C_i of the n-dimensional quantity space. The necessary and sufficient conditions for the existence of an indicator of the second de-*

gree, for which the straight lines, $C_1, \cdots, C_k$ become Engel curves in the periods $t_1, \cdots, t_k$, can be formulated as follows:

(1) The integrability conditions (15) must be fulfilled.

(2) The quadratic forms, $Q^{(1)}, Q^{(2)}, \cdots, Q^{(2k-1)}$, must be negative definite or semidefinite.

If the conditions (1) and (2) are fulfilled, there exists only one indicator of the second degree (with the exception of a proportionality factor and additive constant) in the linear space determined by $C_1, \cdots, C_k$, for which $C_1, \cdots, C_k$ become Engel curves.

It should still be remarked that if the second differential of $f(\lambda_1, \cdots, \lambda_{2k-1})$ that is to say, if $Q=\sum_{\mu=1}^{2k-1}\sum_{\nu=1}^{2k-1}\alpha_{\mu,\nu}d\lambda_\mu d\lambda_\nu$ is a negative or semidefinite form, then all the quadratic forms, $Q^1, \cdots, Q^{(2k-1)}$, are also negative definite or semidefinite.

In our numerical example given above, k is equal to 2. Hence we have to consider the quadratic forms $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$. From (22) and (23) follows immediately, that $Q^{(2)}$ is identical with $Q^{(3)}$. Hence we consider only $Q^{(1)}$ and $Q^{(2)}$. From (16), (17), and (22) we get:

$$\overset{(1)}{\alpha}_{1,1} = \alpha_{1,1} - 2\frac{\rho_{1,1}}{\rho_{1,3}}\alpha_{1,3} + \left(\frac{\rho_{1,1}}{\rho_{1,3}}\right)^2\alpha_{3,3} = -1.6736,$$

$$\overset{(1)}{\alpha}_{1,2} = \alpha_{1,2} - \frac{\rho_{1,2}}{\rho_{1,3}}\alpha_{1,3} - \frac{\rho_{1,1}}{\rho_{1,3}}\alpha_{2,3} + \frac{\rho_{1,2}\rho_{1,1}}{(\rho_{1,3})^2}\alpha_{3,3} = 0.3719,$$

$$\overset{(1)}{\alpha}_{2,2} = \alpha_{2,2} - 2\frac{\rho_{1,2}}{\rho_{1,3}}\alpha_{2,3} + \left(\frac{\rho_{1,2}}{\rho_{1,3}}\right)^2\alpha_{3,3} = -0.0827,$$

$$\overset{(2)}{\alpha}_{1,1} = \alpha_{1,1} - 2\frac{\rho_{2,1}}{\rho_{2,3}}\alpha_{1,3} + \left(\frac{\rho_{2,1}}{\rho_{2,3}}\right)^2\alpha_{3,3} = -2.53125,$$

$$\overset{(2)}{\alpha}_{1,2} = \alpha_{1,2} - \frac{\rho_{2,2}}{\rho_{2,3}}\alpha_{1,3} - \frac{\rho_{2,1}}{\rho_{2,3}}\alpha_{2,3} + \frac{\rho_{2,2}\rho_{2,1}}{(\rho_{2,3})^2}\alpha_{3,3} = 0.5625$$

$$\overset{(2)}{\alpha}_{2,2} = \alpha_{2,2} - 2\frac{\rho_{2,2}}{\rho_{2,3}}\alpha_{2,3} + \left(\frac{\rho_{2,2}}{\rho_{2,3}}\right)^2\alpha_{3,3} = -0.125.$$

We see that

$$\overset{(1)}{\alpha}_{1,1} < 0, \quad \begin{vmatrix} \overset{(1)}{\alpha}_{1,1}, & \overset{(1)}{\alpha}_{1,2} \\ \overset{(1)}{\alpha}_{1,2}, & \overset{(1)}{\alpha}_{2,2} \end{vmatrix} > 0, \quad \overset{(2)}{\alpha}_{1,1} < 0, \text{ and } \begin{vmatrix} \overset{(2)}{\alpha}_{1,1}, & \overset{(2)}{\alpha}_{1,2} \\ \overset{(2)}{\alpha}_{1,2}, & \overset{(2)}{\alpha}_{2,2} \end{vmatrix} = 0.$$

Hence $Q^{(1)}$ is negative definite and $Q^{(2)}$ is negative semidefinite. Thus in this case the sufficient conditions for equilibrium are also fulfilled along the Engel curves C_1 and C_2.

Finally let us make some remarks about the values of the marginal utility of money, which we got in our numerical example. We put $w_0=1$, and for w_1, w_2, and w_3 we get the values 2.5, 5/3, 2.5, where w_0, w_1 denote the marginal utility of money in the period 1 at the incomes 0 and 3, and w_2, w_3 denote the marginal utility of money in the period 2 at the incomes 3 and 6. We see that in both periods the marginal utility of money increases with increasing income. This does not mean any contradiction. It can easily be shown that for any system of indifference surfaces an indicator can be constructed such that the marginal utility of money increases with increasing income. In fact, if $I(q)$ is an indicator, then every monotonically increasing function $F(I)$ is also an indicator. The marginal utility of money corresponding to the indicator $F(I)$ is $F'(I)$ times the marginal utility of money corresponding to the indicator $I(q)$, where $F'(I)$ can be equal to an arbitrary positive function. If $F'(I)$ increases rapidly enough with increasing I, then the marginal utility of money corresponding to the indicator $F(I)$ will also increase with increasing I.

Let us examine generally the question in what cases we shall get an increasing marginal utility of money with increasing income, when the indicator is a polynomial of the second degree in q. We consider the two points q_i and q_{i+1} which we have chosen on the Engel curve C_j $(i=2j-2)$. According to (9) we have:

$$(24) \qquad w_i - w_{i+1} = \frac{w_1\rho_{1,i} - \rho_{0,i} + \rho_{0,1}}{\rho_{i,1}} - \frac{w_1\rho_{1,i+1} - \rho_{0,i+1} + \rho_{0,1}}{\rho_{i+1,1}}.$$

Since q_i and q_{i+1} lie on the same Engel curve, we have $p_i=p_{i+1}$ and therefore $\rho_{i,1}=\rho_{i+1,1}$. On account of the fact that $p_0=p_1$, we have also $\rho_{1,i}=\rho_{0,i}$ and $\rho_{1,i+1}=\rho_{0,i+1}$. Hence

$$(24^*) \qquad \begin{aligned} w_i - w_{i+1} &= \frac{w_1(\rho_{1,i} - \rho_{1,i+1}) + \rho_{1,i+1} - \rho_{1,i}}{\rho_{i,1}} \\ &= \frac{1 - w_1}{\rho_{i,1}}(\rho_{1,i+1} - \rho_{1,i}). \end{aligned}$$

Let us make the assumption:

(α) If $\bar{q}$ and $\bar{\bar{q}}$ are two points of the Engel curve C_r $(r=1, 2, \cdots, k)$ such that the value of $(\bar{q}-\bar{\bar{q}})$, calculated with the prices of the period r, is positive, then the value of $(\bar{q}-\bar{\bar{q}})$, calculated with the prices of any other periods s $(s=1, 2, \cdots, k)$, is also positive. This assumption will be fulfilled in most practical cases.

It is obvious that $\rho_{1,i+1}-\rho_{1,i}=p_1(q_{i+1}-q_i)$. On account of the assumption (α) the expression $p_1(q_{i+1}-q_i)$ has the same sign as $E_{i+1}-E_i$,

and $\rho_{i,1}$ has the same sign as E_1-E_0, where E_0, E_1, E_i, E_{i+1} denote the incomes corresponding to the points q_0, q_1, q_i, q_{i+1}. Hence:

$$\text{(25)} \qquad \text{sign } (w_i - w_{i+1}) = \text{sign } \frac{1 - w_1}{E_1 - E_0} (E_{i+1} - E_i).$$

Now we shall show that the expression $(1-w_1)/(E_1-E_0)$ is positive if the marginal utility of money decreases in the period 1 with increasing income. In fact, if $E_1-E_0>0$, then $w_1<w_0=1$, and therefore $1-w_1$ is also positive. If $E_1-E_0<0$, then $w_1>w_0=1$, and therefore $1-w_1$ is also negative. Hence the ratio $(1-w_1)/(E_1-E_0)$ is always positive. It can be shown analogously that $(1-w_1)/(E_1-E_0)$ is always negative if the marginal utility of money increases in the period 1 with increasing income. Hence we get from (25): sign (w_i-w_{i+1}) is equal to sign $(E_{i+1}-E_i)$ if the marginal utility of money decreases in the period 1 with increasing income, and sign $(w_i-w_{i+1})=$sign (E_i-E_{i+1}) if the marginal utility of money increases in the period 1 with increasing income. In other words: the marginal utility of money decreases (increases) in the period r $(r=2, 3, \cdots, k)$ with increasing income, if the marginal utility of money decreases (increases) in the period 1 with increasing income.

II. THE INTEGRABILITY CONDITIONS AND THE INDICATOR EXPRESSED IN TERMS OF THE ENGEL CURVES

Since the indicator $I(q)$ is a polynomial of the second degree, the Engel curves must be linear. Let us represent the Engel curve C_i $(i=1, 2, \cdots, k)$ of the period t_i by the following equations:

$$\text{(26)} \quad q^1 = \alpha_i^1 E_i + \beta_i^1, \quad q^2 = \alpha_i^2 E_i + \beta_i^2, \quad \cdots, q^n = \alpha_i^n E_i + \beta^n{}_i,$$

where α_i^j, β_i^j denote some constants and E_i denotes the income in the period t_i. The quantities $\alpha_i^1, \alpha_i^2, \cdots, \alpha_i^n$ we shall call the slopes of the Engel curve C_i.

In Section I we have expressed the integrability conditions and the indicator by means of the $2k$ points $q_0, q_1, \cdots, q_{2k-1}$ chosen on the k Engel curves $C_1, \cdots, C_k$. But in fact neither the indicator nor the integrability conditions depend on the special choice of the points $q_0, q_1, \cdots, q_{2k-1}$. In this section we will give a representation of the indicator and of the integrability conditions in terms of the Engel curves, that is to say, in terms of the constants α_i^j, β_i^j.

We would be able to get the desired representation also by substituting in the formulae of the previous section for the co-ordinates

$q_i^1, q_i^2, \cdots, q_i^n$ of the point q_i their expressions from (26). But this would lead to complicated calculations, and we prefer to deduce the desired representation by other more simple considerations.

Let us introduce the following notations:

$$\sum_{r=1}^{n} \alpha_i^r \bar{p}_j^r = a_{i,j} \quad \text{and} \quad \sum_{r=1}^{n} \beta_i^r \bar{p}_j^r = b_{i,j}, \tag{27}$$

where $\bar{p}_j = (\bar{p}_j^1, \cdots, \bar{p}_j^n)$ denotes the system of prices in the period t_j.

It is obvious that

$$a_{i,i} = 1 \quad \text{and} \quad b_{i,i} = 0, \qquad (i = 1, 2, \cdots, k). \tag{28}$$

For the sake of simplification we will assume that

$$a_{i,j} \neq 0, \quad (i = 1, 2, \cdots, k; j = 1, 2, \cdots, k) \quad \text{and} \quad a_{1,2}a_{2,1} \neq 1. \tag{29}$$

Such an assumption practically does not mean any restriction, because by arbitrary small changes in the empirical data the validity of the inequalities (29) can be obtained.

First we shall prove some propositions.

1. *The marginal utility of money is a linear function of the income along each Engel curve C_i $(i=1, 2, \cdots, k)$, when the indicator is a polynomial of the second degree.*

Along the Engel curve C_i the following equations hold true:

$$\frac{\partial I}{\partial q^1} - w_i(E_i)\bar{p}_i^1 = \cdots = \frac{\partial I}{\partial q^n} - w_i(E_i)\bar{p}_i^n = 0,$$

$$\sum_{r=1}^{n} q^r \bar{p}^r = E_i.$$

where $w_i(E_i)$ denotes the marginal utility of money in the period t_i.

Since $I(q)$ is a polynomial of the second degree, $\partial I/\partial q^1, \cdots, \partial I/\partial q^n$ are linear functions of $q^1, \cdots, q^n$. Hence the above equations are linear in the unknown quantities $q^1, q^2, \cdots, q^n, w_i$. Since the right-hand side of these equations are linear functions of E_i, the solutions must also be linear functions of E_i. Hence proposition 1 is proved.

2. *The marginal utility of money $w_i(E_i)$ along the Engel curve C_i $(i=1, \cdots, k)$ is not constant.*

First we shall show that $w_1(E_1)$ is not constant. Let us denote two different points of C_1 by q_0 and q_1, and two different points of C_2 by q_2 and q_3. Denote the incomes corresponding to q_0, q_1, q_2, q_3 by $E_1^0, E_1^1, E_2^2,$

E_2^3. We put $w_1(E_1^0)=1$.[3] We have to show that $w_1(E_1^1)\neq 1$. According to (10) we have

$$w_1=\frac{\rho_{0,2}-\rho_{0,3}+\rho_{2,3}\dfrac{\rho_{0,1}-\rho_{0,2}}{\rho_{2,1}}-\rho_{3,2}\dfrac{\rho_{0,1}-\rho_{0,3}}{\rho_{3,1}}}{\rho_{3,2}\dfrac{\rho_{1,3}}{\rho_{3,1}}-\rho_{2,3}\dfrac{\rho_{1,2}}{\rho_{2,1}}},$$

where

$$\begin{aligned}
\rho_{0,1}=\rho_{1,1}&=\sum \bar{p}_1(q_1-q_0)=E_1^1-E_1^0,\\
\rho_{0,2}=\rho_{1,2}&=\sum \bar{p}_1(q_2-q_0)=a_{2,1}E_2^2+b_{2,1}-a_{11}E_1^0-b_{1,1}\\
&=a_{2,1}E_2^2+b_{2,1}-E_1^{(0)},\\
\rho_{0,3}=\rho_{1,3}&=\sum \bar{p}_1(q_3-q_0)=a_{2,1}E_2^3+b_{2,1}-E_1^0,\\
\rho_{2,1}=\rho_{3,1}&=\sum \bar{p}_2(q_1-q_0)=a_{1,2}(E_1^1-E_1^0),\\
\rho_{2,2}=\rho_{3,2}&=\sum \bar{p}_2(q_2-q_0)=E_2^2-a_{1,2}E_1^0-b_{1,2},\\
\rho_{2,3}=\rho_{3,3}&=\sum \bar{p}_2(q_3-q_0)=E_2^3-a_{1,2}E_1^0-b_{1,2}.
\end{aligned}$$

If we substitute in the right-hand side of the above equation for $\rho_{i,j}$ their expressions given here, then we get:

$$\begin{aligned}
w_1&=-\frac{a_{2,1}(E_2^2-E_2^3)+\dfrac{1}{a_{1,2}(E_1^1-E_1^0)}\cdot[\rho_{0,1}(\rho_{2,3}-\rho_{3,2})+\rho_{3,2}\rho_{0,3}-\rho_{2,3}\rho_{0,2}]}{\dfrac{1}{a_{1,2}(E_1^1-E^0)}\cdot(\rho_{3,2}\rho_{0,3}-\rho_{2,3}\rho_{0,2})}\\
&=\frac{a_{1,2}a_{2,1}(E_2^2-E_2^3)(E_1^1-E_1^0)+\rho_{0,1}(\rho_{2,3}-\rho_{3,2})}{\rho_{3,2}\rho_{0,3}-\rho_{2,3}\rho_{0,2}}+1\\
&=\frac{a_{1,2}a_{2,1}(E_2^2-E_2^3)(E_1^1-E_1^0)+(E_2^3-E_2^2)(E_1^1-E_1^0)}{\rho_{3,2}\rho_{0,3}-\rho_{2,3}\rho_{0,2}}+1\\
&=\frac{(a_{1,2}a_{2,1}-1)(E_2^2-E_2^3)(E_1^1-E_1^0)}{\rho_{3,2}\rho_{0,3}-\rho_{2,3}\rho_{0,2}}+1.
\end{aligned}$$

Since $q_1\neq q_0$ and $q_2\neq q_3$, we have $E_2^2-E_2^3\neq 0$ and $E_1^1-E_1^0\neq 0$. On account of (29) we also have $a_{1,2}a_{2,1}-1\neq 0$, hence $w_1\neq 1$.

We have now to prove that $w_i(E_i)$ is not constant for $i=2, 3, \cdots, k$.

[3] In one point of the quantity space we can choose arbitrarily the value of the marginal utility of money. In fact, by determining the value $w_1(E_1^{(0)})$ we have only determined the proportionality factor of the indicator.

Let us consider two different points q_{2i-2} and q_{2i-1} of the Engel curve C_i corresponding to the incomes $E_i^{(2i-2)}$ and $E_i^{(2i-1)}$. Substituting in (24*) $2i-2$ for i, we get:

$$w_i(E_i^{(2i-2)}) - w_i(E_i^{(2i-1)}) = \frac{1-w_1}{\rho_{2i-2,1}}(\rho_{1,2i-1} - \rho_{1,2i-2}),$$

where

$$\rho_{2i-2,1} = \sum \bar{p}_i(q_1 - q_0) = a_{1,i}E_1^1 + b_{1,i} - a_{1,i}E_1^0 - b_{1,i} = a_{1,i}(E_1^1 - E_1^0),$$
$$\rho_{1,2i-1} = \sum \bar{p}_1(q_{2i-1} - q_0) = a_{i,1}E_i^{(2i-1)} + b_{i,1} - E_1^{(0)},$$
$$\rho_{1,2i-2} = \sum \bar{p}_1(q_{2i-2} - q_0) = a_{i,1}E_i^{(2i-2)} + b_{i,1} - E_1^0.$$

Substituting in the above equation for $\rho_{\mu,\nu}$ their expressions given here, we get

$$w_i(E_i^{(2i-2)}) - w_i(E_i^{(2i-1)}) = \frac{1-w_1}{a_{1,i}(E_1^1 - E_1^0)} a_{i,1}(E_i^{(2i-1)} - E_i^{(2i-2)}).$$

Since $w_1 \neq 1$, $a_{i,1} \neq 0$, and $E_i^{(2i-2)} - E_i^{(2i-1)} \neq 0$, the right-hand side of this equation is not equal to zero. Hence the proposition 2 is proved.

From the propositions 1 and 2 we get easily

3. *In each time period t_i $(i=1, 2, \cdots, k)$ there exists one and only one income E_i^*, at which the marginal utility of money is equal to zero.*[4]

If we denote the point of C_i corresponding to the income E_i^* by $\hat{q}_i$ $(i=1, 2, \cdots, k)$, then the following proposition can be proved:

4. *The linear space S^* determined by the points $\hat{q}_1, \cdots, \hat{q}_k$, is an iso-value space relative to the price systems $\bar{p}_1, \cdots, \bar{p}_k$, that is to say, for any two points $r=(r^1, \cdots, r^n)$ and $s=(s^1, \cdots, s^n)$ of S^* the equation $\sum_{\mu=1}^n \bar{p}_j^\mu r^\mu = \sum_{\mu=1}^n \bar{p}_j^\mu s^\mu$ $(j=1, 2, \cdots, k)$ holds true.*

Since the marginal utility of money is equal to zero at the point $\hat{q}$ $(i=1, 2, \cdots, k)$, the marginal utility of each good must also be equal to zero. Hence the marginal utility of each good is equal to zero at every point of S^*. From this fact follows, that the indicator $I(q)$ is constant in S^*. It can easily be shown that the indicator $I(q)$ must be constant also in every linear space $\overline{S}^*$ parallel to S^* and of the dimension less than or equal to the dimension of S^*.[5] Let us consider a point

[4] E_i^* will not lie, of course, in the region T of the quantity space, where the approximation of the indicator by a polynomial of the second degree is valid. But this fact does not alter our considerations.

[5] This can be proved as follows: Since $\partial I(q)/\partial q^1 = \cdots = \partial I(q)/\partial q^n = 0$ for each point q of S^*, the linear terms in $I(q)$ will disappear in every linear co-ordinate system whose origin lies in S^*. Denote the new co-ordinates by $u^1, \cdots, u^n$, where $u^1, \cdots, u^n$ are certain linear functions of $q^1, \cdots, q^n$; then the indicator ex-

q_i of the Engel curve C_i where the marginal utility of money is positive. Denote by $\overline{S}^*$ the linear space through q_i which is parallel to S^* and has the same dimension as S^*. Since $I(q)$ is constant in $\overline{S}^*$ and the marginal utility of money is >0 in q_i, the expression $\sum_{\mu=1}^{n} \bar{p}_i^{\mu}(q_i^{\mu} - s^{\mu})$ must be equal to zero for each point $s=(s^1, \cdots, s^n)$ of $\overline{S}^*$. Hence for any two points r and s of $\overline{S}^*$ the expression $\sum_{\mu=1}^{n} \bar{p}_i^{\mu}(r^{\mu} - s^{\mu}) = 0$, that is to say, $\overline{S}^*$ is an isovalue space relative to the system of prices $\bar{p}_i$. Since $\overline{S}^*$ is parallel to S^*, S^* must also be an isovalue space relative to $\bar{p}_i$ $(i=1, 2, \cdots, k)$. Hence the proposition 4 is proved.

We will now calculate the incomes $E_1^*, \cdots, E_k^*$. On account of proposition 4 we have the following equations:

$$a_{1,j}E_1^* + b_{1,j} = a_{i,i}E_i^* + b_{i,j} \qquad \begin{matrix}(i = 2, \cdots, k),\\ (j = 1, 2, \cdots, k).\end{matrix} \tag{30}$$

For $j=1$ we get:

$$a_{1,1}E_1^* + b_{1,1} = E_1^* = a_{i,1}E_i^* + b_{i,1}.$$

Hence

$$E_i^* = \frac{E_1^* - b_{i,1}}{a_{i,1}}. \tag{31}$$

From (30) and (31) we get:

$$a_{1,j}E_1^* + b_{1,j} = a_{i,j}\frac{E_1^* - b_{i,1}}{a_{i,1}} + b_{i,j} \qquad \begin{matrix}(i = 2, \cdots, k),\\ (j = 2, \cdots, k).\end{matrix} \tag{32}$$

The equations (32) are equivalent to the equations:

$$\begin{aligned} a_{1,i}E_1^* + b_{1,i} = a_{i,i}\frac{E_1^* - b_{i,1}}{a_{i,1}} + b_{i,i} &= \frac{E_1^* - b_{i,1}}{a_{i,1}}, \qquad (i = 2, \cdots, k),\\ a_{1,j}E_1^* + b_{1,j} = a_{i,j}\frac{E_1^* - b_{i,1}}{a_{i,1}} + b_{i,j}, &\qquad (i = 2, \cdots, k, j = 2, \cdots, k, i \neq j). \end{aligned} \tag{33}$$

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pressed as a function of $u^1, \cdots, u^n$ has the form: $I(q) = \sum_{i=1}^{n}\sum_{i=1}^{n} g_{i,i}u^i u^i + g$, where $g_{i,i}$ and g denote certain constants. The quadratic form $\sum\sum g_{i,i}u^i u^i$ can be transformed in a sum of squares, that is to say, $\sum\sum g_{i,i}u^i u^i \equiv [L_1(u_1, \cdots, u_n)]^2 + \cdots + [L_s(u_1, \cdots, u_n)]^2$, where $s \leqq n$ and $L_1, \cdots, L_s$ are linearly independent linear forms. Since $\partial I/\partial u^1 = \cdots = \partial I/\partial u^n = 0$ in every point of S^*, and since $L_1, \cdots, L_s$ are linearly independent, we have $L_1 = \cdots = L_s = 0$ at each point of S^*, that is to say, S^* is contained in the linear space given by the equations $L_1 = L_2 = \cdots = L_s = 0$. Hence a linear space $\overline{S}$ parallel to S^* is contained in the linear space L given by the equations $L_1 = a_1, \cdots, L_s = a_s$ where $a_1, \cdots, a_s$ denote certain constants. In the linear space L the indicator has the constant value $a_1^2 + \cdots + a_s^2 + g$ and therefore also in $\overline{S}$.

From the first equation of (33) we get

$$E_1^* = \frac{a_{i,1}b_{1,i} + b_{i,1}}{1 - a_{1,i}a_{i,1}}, \qquad (i = 2, \cdots, k). \tag{34}$$

Hence on account of (31) we have

$$\begin{aligned} E_i^* &= \frac{1}{a_{i,1}}\left(\frac{a_{i,1}b_{1,i} + b_{i,1}}{1 - a_{1,i}a_{i,1}} - b_{i,1}\right) = \frac{1}{a_{i,1}} \cdot \left(\frac{a_{i,1}b_{1,i} + a_{1,i}a_{i,1}b_{i,1}}{1 - a_{1,i}a_{i,1}}\right) \\ &= \frac{b_{1,i} + a_{1,i}b_{i,1}}{1 - a_{1,i}a_{i,1}}, \qquad (i = 2, \cdots, k). \end{aligned} \tag{35}$$

If we calculate E_1^* from the second equation of (33) we get

$$E_1^* = \frac{b_{i,j} - b_{1,j} - \dfrac{a_{i,j}b_{i,1}}{a_{i,1}}}{a_{1,j} - \dfrac{a_{i,j}}{a_{i,1}}} = \frac{\dfrac{a_{i,1}}{a_{i,j}}(b_{i,j} - b_{1,j}) - b_{i,1}}{\dfrac{1}{a_{i,j}}(a_{i,1}a_{1,j} - a_{i,j})}. \tag{36}$$

On account of (34) and (36) we see that the elimination of E_1^* from (33) leads to the following system of equations:

$$\frac{a_{i,1}b_{1,i} + b_{i,1}}{a_{i,1}a_{1,i} - 1} = \frac{a_{j,1}b_{1,j} + b_{j,1}}{a_{j,1}a_{1,j} - 1}, \tag{37}$$

$$\frac{a_{i,1}b_{1,i} + b_{i,1}}{1 - a_{i,1}a_{1,i}} = \frac{\dfrac{a_{i,1}}{a_{i,j}}(b_{i,j} - b_{1,j}) - b_{i,1}}{\dfrac{1}{a_{i,j}}(a_{i,1}a_{1,j} - a_{i,j})}, \tag{38}$$

$$\frac{\dfrac{a_{i,1}}{a_{i,j}}(b_{i,j} - b_{1,j}) - b_{i,1}}{\dfrac{1}{a_{i,j}}(a_{i,1}a_{1,j} - a_{i,j})} = \frac{\dfrac{a_{j,1}}{a_{j,i}}(b_{j,i} - b_{1,i}) - b_{j,1}}{\dfrac{1}{a_{j,i}}(a_{j,1}a_{1,i} - a_{j,i})}, \tag{39}$$

where $i<j$, $i=2, 3, \cdots, k$, and $j=2, 3, \cdots, k$.

The equations (37), (38), and (39) are necessary and sufficient for the existence of an income E_i^* in each time period t_i $(i=1, \cdots, k)$ such that the linear space S^*, determined by the points $\hat{q}_1, \cdots, \hat{q}_k$ of the Engel curves $C_1, \cdots, C_k$ corresponding to the incomes $E_1^*, \cdots, E_k^*$, is an isovalue space relative to the price systems $\bar{p}_1, \cdots, \bar{p}_k$.

In Section I we have chosen two points q_{2i-2} and q_{2i-1} on each Engel curve C_i $(i=1, \cdots, k)$. In order to get the integrability conditions in terms of the Engel curves, we shall put $q_{2i-2}=\hat{q}_i$, where $\hat{q}_i$ denotes the point of C_i corresponding to the income E_i^*. The point $q_{2i-1} \neq q_{2i-2}$ can

be chosen arbitrarily. Denote the income corresponding to q_{2i-1} by $\overline{E}_i$. Since the linear space S^* determined by the points $\hat{q}_1, \cdots, \hat{q}_k$ is an isovalue space, we get for every even integer $j=2r<2k$:

$$\rho_{i,j} = \sum p_i(q_j - q_0) = \sum p_i(q_{2r} - q_0) = \sum p_i(\hat{q}_{r+1} - \hat{q}_1) = 0, \quad (j \text{ even}), \tag{40}$$

where p_i denotes the price system corresponding to the Engel curve on which q_i lies. If $j=2r+1$, then we have

$$\begin{aligned} \rho_{i,j} &= \sum p_i(q_{2r+1} - q_0) = \sum p_i[(q_{2r+1} - q_{2r}) + (q_{2r} - q_0)] \\ &= \sum \bar{p}_s(q_{2r+1} - q_{2r}) = a_{r+1,s}(\overline{E}_{r+1} - E^*_{r+1}) \end{aligned} \tag{41}$$

where $s=\frac{1}{2}(i+1)$ if i is odd, and $s=\frac{1}{2}(i+2)$ if i is even.

The equations (37), (38), and (39) represent only a part of the integrability conditions. In order to get all integrability conditions, we consider the equations (7). Since w_i and $\rho_{i,j}$ are equal to zero for all even values of j (zero is also considered as an even value), the equations (7) become $0=0$, if at least one of the integers i, j is even. Hence we have to consider only the case where both integers i and j are odd. Denote i by $2r-1$ and j by $2s-1$; then the equations (7) become:

$$w_r(\overline{E}_r)\rho_{2r-1,2s-1} = w_s(\overline{E}_s)\rho_{2s-1,2r-1}, \quad \begin{matrix} (r = 1, \cdots, k), \\ (s = 1, \cdots, k), \end{matrix}$$

where $w_\mu(\overline{E}_\mu)$ denotes the marginal utility of money in the period μ at the income $\overline{E}_\mu$ $(\mu=1, \cdots, k)$. If we substitute in the above equation for $\rho_{i,j}$ its expression given by (41), then we get

$$a_{s,r}(\overline{E}_s - E^*_s)w_r(\overline{E}_r) = a_{r,s}(\overline{E}_r - E^*_r)w_s(\overline{E}_s). \tag{42}$$

For $r=1$ we get

$$a_{s,1}(\overline{E}_s - E^*_s)w_1(\overline{E}_1) = a_{1,s}(\overline{E}_1 - E^*_1)w_s(\overline{E}_s).$$

Since in one point of the quantity space we can arbitrarily choose the marginal utility of money, we put $w_1(\overline{E}_1)=1$. Then we get from the above equation

$$w_s(\overline{E}_s) = \frac{a_{s,1}}{a_{1,s}} \frac{\overline{E}_s - E^*_s}{\overline{E}_1 - E^*_1}, \qquad (s = 2, \cdots, k). \tag{43}$$

From (42) we get, on account of (43),

$$a_{sr}(\overline{E}_s - E^*_s) \frac{a_{r,1}}{a_{1,r}} \frac{\overline{E}_r - E^*_r}{\overline{E}_1 - E^*_1} = a_{r,s}(\overline{E}_r - E^*_r) \frac{a_{s,1}}{a_{1,s}} \frac{\overline{E}_s - E^*_s}{\overline{E}_1 - E^*_1}.$$

Hence

$$\frac{a_{s,r}a_{r,1}}{a_{1,r}} = \frac{a_{r,s}a_{s,1}}{a_{1,s}}. \tag{44}$$

On account of (44), the denominators in (39) are equal to each other. Hence (39) can be written as follows:

$$\frac{a_{i,1}}{a_{i,j}}(b_{i,j}-b_{1,j})-b_{i,1}=\frac{a_{j,1}}{a_{j,i}}(b_{j,i}-b_{1,i})-b_{j,1}. \tag{45}$$

The equations (37), (38), (44), and (45) represent the whole system of integrability conditions.

We can summarize our results in the following

THEOREM III: *The necessary and sufficient conditions of integrability—when the indicator is a polynomial of the second degree—are given by the following four systems of equations:*

$$\frac{a_{i,j}a_{j,1}}{a_{1,j}}=\frac{a_{j,i}a_{i,1}}{a_{1,i}}, \tag{46}$$

$$\frac{a_{i,1}b_{1,i}+b_{i,1}}{a_{i,1}a_{1,i}-1}=\frac{a_{j,1}b_{1,j}+b_{j,1}}{a_{j,1}a_{1,j}-1}, \tag{47}$$

$$\frac{a_{i,1}b_{1,i}+b_{i,1}}{1-a_{i,1}a_{1,i}}=\frac{\dfrac{a_{i,1}}{a_{i,j}}(b_{i,j}-b_{1,j})-b_{i,1}}{\dfrac{1}{a_{i,j}}(a_{i,1}a_{1,j}-a_{i,j})}, \tag{48}$$

$$\frac{a_{i,1}}{a_{i,j}}(b_{i,j}-b_{1,j})-b_{i,1}=\frac{a_{j,1}}{a_{j,i}}(b_{j,i}-b_{1,i})-b_{j,1}, \tag{49}$$

for all integers i and j for which $2\leqq i<j$ and $j\leqq k$. The quantities $a_{i,j}$ and $b_{i,j}$ are defined by (27).

Since only two variable indices appear in the conditions of integrability, namely i and j, we can deduce the following important

THEOREM IV. *If a system of Engel curves $C_1, \cdots, C_k$—under a second-degree polynomial indicator—are given and each set of three Engel curves satisfies the integrability conditions, then the system of the Engel curves $C_1, \cdots, C_k$ as a whole, satisfies also the integrability conditions.*

From well-known propositions of analytical geometry the following theorems can be easily deduced:

THEOREM V. *Each point of the isovalue space S^* is center of the indifference surfaces.*

THEOREM VI. *If r is the smallest integer such that the indicator $I(q)$ can be transformed into a sum of r squares, that is to say, if the rank of the quadratic form $I(q)$ is equal to r, then the dimension of S^* is equal to $n-r$, where n denotes the dimension of the quantity space.*

THEOREM VII. *If the rank of the quadratic form $I(q)$ is equal to n, then all Engel curves have a common point of intersection and S^* will be reduced to this single point.*

Since the equations (37), (38), and (39) are certainly fulfilled, if all Engel curves have a common point of intersection, we get the

THEOREM VIII. *If all Engel curves have a common point of intersection, then this point will be the center of the indifference surfaces and the integrability conditions will be reduced to the single system of equations:*

$$\frac{a_{i,j}a_{j,1}}{a_{1,j}} = \frac{a_{j,i}a_{i,1}}{a_{1,i}}, \qquad (2 \leqq i < j).$$

Now let us represent the indicator in terms of the Engel curves. For this purpose we choose two points q_{2i-2} and q_{2i-1} on each Engel curve C_i $(i=1, \cdots, k)$ such that the income corresponding to q_{2i-2} is equal to E_i^* and the income corresponding to q_{2i-1} is equal to $\overline{E}_i = E_i^* + 1$. The marginal utility of money in the period 1 at the income $E_1^* + 1$ we shall put equal to 1. The system of vectors, $v_1 = q_0q_1, \cdots, v_{2k-1} = q_0q_{2k-1}$, we call the distinguished system of vectors and the vector co-ordinates of a point q relative to this system we call the distinguished vector co-ordinates of q.

Denote the distinguished vector co-ordinates of a point q by $\lambda_1, \cdots, \lambda_{2k-1}$ and the indicator expressed as a function of $\lambda_1, \cdots, \lambda_{2k-1}$ by $f(\lambda_1, \cdots, \lambda_{2k-1})$. Then we have according to (6)

$$\frac{\partial^2 f}{\partial\lambda_i\partial\lambda_j} = w_j\rho_{j,i} - w_0\rho_{0,i}, \qquad \begin{matrix}(i = 1, \cdots, 2k-1),\\ (j = 1 \cdots, 2k-1),\end{matrix}$$

where w_μ denotes the marginal utility of money at the point q_μ $(\mu = 0, 1, \cdots, 2k-1)$. Since $w_j = 0$ if j is even, and $\rho_{j,i}$ is equal to zero if i is even, we get from the above equation

$$\frac{\partial^2 f}{\partial\lambda_i\partial\lambda_j} = 0, \text{ if one of the integers } i \text{ and } j \text{ is even,} \tag{49'}$$

and

$$\frac{\partial^2 f}{\partial\lambda_{2r-1}\partial\lambda_{2s-1}} = w_s(\overline{E}_s)\rho_{2s-1,2r-1}, \tag{50}$$

where $w_s(\overline{E}_s)$ denotes the marginal utility of money in the period s at the income $\overline{E}_s = E_s^* + 1$, that is to say, at the point q_{2s-1}.

From (50) we get, on account of (41) and (43),

$$\frac{\partial^2 f}{\partial\lambda_{2r-1}\partial\lambda_{2s-1}} = \frac{a_{s,1}}{a_{1,s}}\frac{\overline{E}_s - E_s^*}{\overline{E}_1 - E_1^*}a_{r,s}(\overline{E}_r - E_r^*) = \frac{a_{s,1}a_{r,s}}{a_{1,s}}. \tag{51}$$

Since the marginal utility of money is equal to zero at the point q_0, we have

$$\frac{\partial f(0, 0, \cdots, 0)}{\partial \lambda_i} = 0, \qquad (i = 1, 2, \cdots, k). \tag{52}$$

On account of (49′), (51), and (52) we get the following

THEOREM IX. *The indicator $I(q)$ expressed as a function of the distinguished co-ordinates $\lambda_1, \cdots, \lambda_{2k-1}$ of the point q is given by the following formula*

$$I(q) = f(\lambda_1, \cdots, \lambda_{2k-1}) = \tfrac{1}{2} \sum_{s=1}^{k} \sum_{r=1}^{k} \frac{a_{s,1} a_{r,s}}{a_{1,s}} \lambda_{2r-1} \lambda_{2s-1}. \tag{53}$$

If we want to have the indicator expressed as a function of the Cartesian co-ordinates $q^1, q^2, \cdots, q^n$ of the point q, then we calculate $\lambda_1, \cdots, \lambda_{2k-1}$ as functions of $q^1, q^2, \cdots, q^n$ by means of the equations (12) and we substitute in (53) for $\lambda_1, \cdots, \lambda_{2k-1}$ the corresponding expressions.

III. THE SET OF GOODS TO BE CONSIDERED

The number of the different consumers' goods is very large and the calculation of the indicator in the space of all consumers' goods would involve a great amount of computation work. Hence the important question arises whether we can restrict ourselves to the calculation of the indicator in the space of a certain subset of consumers' goods. This can actually be done under certain conditions, which we shall formulate in this section.

Let us divide the set K of all consumers goods nos. $1, 2, \cdots, n$ into three subsets K_1, K_2, K_3; K_1 consisting of the goods nos. $i_1, \cdots, i_r$, K_2 of the goods nos. $j_1, \cdots, j_s$ and K_3 of the goods nos. $k_1, \cdots, k_t$, where the integers $i_1, \cdots, i_r, j_1, \cdots, j_s, k_1, \cdots, k_t$ are different from each other and $r+s+t=n$. The subset K_3 may also be empty, whereas K_1 and K_2 are supposed not to be empty. We say that K_1 is semi-independent of K_2 if the following condition is fulfilled:

If

$$(\bar{q}^{i_1}, \cdots, \bar{q}^{i_r}, \bar{q}^{j_1}, \cdots, \bar{q}^{j_s}, \bar{q}^{k_1}, \cdots, \bar{q}^{k_t}) \text{ and}$$
$$(\bar{\bar{q}}^{i_1}, \cdots, \bar{\bar{q}}^{i_r}, \bar{q}^{j_1}, \cdots, \bar{q}^{j_s}, \bar{q}^{k_1}, \cdots, \bar{q}^{k_t})$$

are two indifferent combination of goods, then also the combinations

$$(\bar{q}^{i_1}, \cdots, \bar{q}^{i_r}, q^{j_1}, \cdots, q^{j_s}, \bar{q}^{k_1}, \cdots, \bar{q}^{k_t}) \text{ and}$$
$$(\bar{\bar{q}}^{i_1}, \cdots, \bar{\bar{q}}^{i_r}, q^{j_1}, \cdots, q^{j_s}, \bar{q}^{k_1}, \cdots, \bar{q}^{k_t})$$

are indifferent, where $q^{j_1}, \cdots, q^{j_s}$ denote arbitrary quantities of the goods nos. $j_1, \cdots, j_s$.

It can easily be shown that if the indicator is a polynomial of the second degree in q the above definition of semi-independence is equivalent to the following one:

The set k_1 is semi-independent of the set k_2 if the marginal utility ratios between the goods of the set k_1, that is to say, if the ratios,

$$\frac{\partial I}{\partial q^{i_2}} \Big/ \frac{\partial I}{\partial q^{i_1}}, \cdots, \frac{\partial I}{\partial q^{i_r}} \Big/ \frac{\partial I}{\partial q^{i_1}},$$

are independent of the quantities of the goods in k_2.

It should be mentioned that the relation of semi-independence is not a symmetrical one. That is to say, if k_1 is semi-independent of k_2, k_2 need not be semi-independent of k_1. Let us illustrate it by an example. We consider 4 consumers' goods, nos. 1, 2, 3, 4. The indicator $I(q)$ is given by the expression

$$I(q) = q^1q^2q^3 - (q^3)^2 + q^3q^4.$$

k_1 consists of the goods nos. 1 and 2, k_2 of the goods nos. 3 and 4, and k_3 is empty. Since the ratio

$$\frac{\partial I}{\partial q^1} \Big/ \frac{\partial I}{\partial q^2} = \frac{q^2}{q^1}$$

does not depend on the goods nos. 3 and 4, k_1 is semi-independent of k_2. But the ratio

$$\frac{\partial I}{\partial q^4} \Big/ \frac{\partial I}{\partial q^3} = \frac{q_3}{q^1q^2 - 2q^3 + q^4}$$

does depend on goods nos. 1 and 2, hence k_2 is not semi-independent of k_1.

The subset k_1 is said to be independent[6] of the subset k_2 if there exists an indicator which is the sum of two functions f_1 and f_2 such that f_1 does not depend on the quantities of the goods in k_2 and f_2 does not depend on the quantities of the goods in k_1.

It should be mentioned that the above definition of semi-independence is much weaker than this definition of independence. This is the reason why we call it semi-independence.

If k_1 is independent of k_2, then k_1 is also semi-independent of k_2, but not inversely. If k_1 is semi-independent of k_2, k_1 need not be independent of k_2. This follows also from the fact that independence is obviously a symmetrical relation, whereas semi-independence is not symmetrical.

It is obvious that if k_1 is semi-independent of the set of all other consumers' goods, then we can speak about indifference surfaces in the

[6] See Ragnar Frisch, ECONOMETRICA, Vol. 4, January, 1936, p. 35.

space of the goods of k_1 without referring to the goods in k_2. Hence we can say:

If a subset k_1 of consumers' goods is semi-independent of the set of all other consumers' goods, then we can restrict ourselves to the consideration of k_1 and all results of the previous section can be well applied to the subset k_1.

This fact enables us to apply the theory to small sets of consumers' goods. It seems very plausible that the set of all foods is not only semi-independent, but also independent of all other consumers' goods. Within the set of foods it will scarcely be possible to form a subset which is independent of all other foods, whereas subsets of foods which are semi-independent of all other foods very probably can be formed. For instance, the set of all sorts of bread will be perhaps semi-independent of all other foods.

IV. SOME APPLICATIONS AND REMARKS

1. *Calculation of the demand functions.* One of the most important problems which can be solved if we know the shape of the indifference surfaces is the determination of the demand functions for consumers' goods, that is to say, the determination of the Engel curve in any given price system.

As we have pointed out in Section I, the indicator is determined only in the linear space S determined by the Engel curves considered. Therefore we shall be able to determine the Engel curve corresponding to a given price system only in case the Engel curve lies in S.

Let us consider k time periods, $t_1, \cdots, t_k$, and the corresponding Engel curves $C_1, \cdots, C_k$. At first we have to formulate the necessary and sufficient conditions which must be satisfied by a price system $p=p^1, \cdots, p^n$ in order that the Engel curve C corresponding to p should lie in the linear space S determined by $C_1, \cdots, C_k$.

We shall call the system of the n marginal utilities $\partial I/\partial q^1, \cdots, \partial I/\partial q^n$ the marginal utility. To each point q of the quantity space belongs a certain marginal-utility vector, which we shall denote by $w(q)$. Since the marginal-utility vector $w(q)$ is a linear function of q, when the indicator is a polynomial of the second degree in q, it is obvious that, for each point q of the linear space determined by $C_1, \cdots, C_k$, $w(q)$ can be represented as a vector sum:

$$w(q) = \sum_{r=0}^{2k-1} \nu_r w(q_r),$$

where q_{2i-2} and q_{2i-1} denote two different points of the Engel curve C_i $(i=1, \cdots, k)$ and $\nu_0, \cdots, \nu_{2k-1}$ are certain numbers.

Since the marginal-utility vector in each point of the Engel curve C_i is proportional to the price vector p_i of the period t_i, the vector sum $\sum \nu_r w(q_r)$ is equal to the vector sum $\sum_{i=1}^{k} k_i p_i$, where $k_1, \cdots, k_k$ denote some numbers. Hence for each point q of S the vector $w(q)$ is a linear combination of the price vectors $p_1, \cdots, p_k$. From this fact follows easily: In order that the Engel curve C corresponding to the price vector p should lie in S, it is necessary and sufficient that the price vector p should be a linear combination of the price vectors $p_1, \cdots, p_k$. Hence we shall consider only the case

$$p = \mu_1 p_1 + \cdots + \mu_k p_k, \tag{54}$$

where $\mu_1, \cdots, \mu_k$ are certain numbers. If k is equal to the dimension of the quantity space and the vectors $p_1, \cdots, p_k$ are linearly independent, then any price vector p can be represented in the form of (54), that is to say, in this case we are able to calculate the Engel curve corresponding to any price vector p.

If the isovalue space S^* (that is, the space where the marginal-utility of money is equal to zero, see p. 163) is not 0-dimensional, then the Engel curve is not uniquely determined by the price vector, but only the projection of the Engel curve on a linear space S' perpendicular to S^* is uniquely determined. The set of all Engel curves corresponding to a certain price vector p, can be obtained as follows: Denote by C one of the possible Engel curves corresponding to p, and denote by q a point of C not lying in S^*. We put through q a linear space $\overline{S}$ parallel to S^* and of the same dimension. The set of all possible Engel curves is identical with the set of all straight lines joining $\overline{S}$ with S^*. We can restrict ourselves to the determination of one of the possible Engel curves.

A point q, for which

$$w(q) = \mu_1 p_1 + \cdots + \mu_k p_k, \tag{55}$$

is certainly a point of an Engel curve corresponding to the price vector $p = \mu_1 p_1 + \cdots + \mu_k p_k$. Hence we will determine the point q for which (55) holds true.

Denote by E_i^* the income in the period t_i for which the marginal utility of money is equal to zero. The value of E_i^* is given by the equations (35) and (36).

We put the marginal utility of money in the period 1 at the income $\overline{E}_1 = E_1^* + 1$ equal to 1. Then we have, according to (43),

$$w_s(\overline{E}_s) = \frac{a_{s,1}}{a_{1,s}} \frac{\overline{E}_s - E_s^*}{\overline{E}_1 - E_1^*} = \frac{a_{s,1}}{a_{1,s}} (\overline{E}_s - E_s^*), \tag{56}$$

where $\overline{E}_s$ denotes the marginal utility of money in the period s at the income $\overline{E}_s$. Let

$$\overline{E}_s = \frac{a_{1,s}}{a_{s,1}}\mu_s + E_s^*,$$

and denote by q_s the point of C_s corresponding to the income $\overline{E}_s$. Then we get on account of (56), that the marginal utility of money at the point q_s is equal to μ_s. Hence

$$w(q_s) = \mu_s p_s. \tag{57}$$

Denote by $\hat{q}_s$ the point of C_s corresponding to the income E_s^* and the vector with the initial point $\hat{q}_s$ and terminal point q_s by v_s $(s=1, \cdots, k)$. Let us translate the vectors $v_1, \cdots, v_k$ to a common initial point q_0 in S^* without changing their directions, and let us denote the translated vectors by $\bar{v}_1, \cdots, \bar{v}_k$. The terminal point $\bar{q}$ of the vectors' sum $\bar{v}=\bar{v}_1+\cdots+\bar{v}_k$ will obviously have the co-ordinates:

$$\begin{aligned}
\bar{q}^1 &= q_0^1 + \sum_{s=1}^{k} \alpha_s^1 \frac{a_{1,s}}{a_{s,1}}\mu_s, \\
\bar{q}^2 &= q_0^2 + \sum_{s=1}^{k} \alpha_s^2 \frac{a_{1,s}}{a_{s,1}}\mu_s, \\
&\cdots\cdots\cdots\cdots \\
\bar{q}^n &= q_0^n + \sum_{s=1}^{k} \alpha_s^n \frac{a_{1,s}}{a_{s,1}}\mu_s,
\end{aligned} \tag{58}$$

where $\alpha_s^1, \cdots, \alpha_s^n$ denote the slopes of the Engel curve C_s.

It can be easily shown, on account of (57) and of the fact that $w(q)=0$ for each point q of S^*, that

$$w(\bar{q}) = \sum_{s=1}^{k} \mu_s p_s.$$

Hence $\bar{q}$ is a point of an Engel curve corresponding to the price vector $p=\sum_{s=1}^{k}\mu_s p_s$. A straight line which joins $\bar{q}$ with a point of S^* is an Engel curve, for instance the straight line C going through $\bar{q}$ and q_0. If we denote the income corresponding to q_0 by E_0^*, then the Engel curve C is given by the equations:

$$q^1 = (E - E_0^*)\frac{\sum_{s=1}^{k} \alpha_s^1 \frac{a_{1,s}}{a_{s,1}}\mu_s}{k} + q_0^1,$$

$$q^2 = (E - E_0^*)\frac{\sum_{s=1}^{k} \alpha_s^2 \frac{a_{1,s}}{a_{s,1}}\mu_s}{k} + q_0^2,$$

. . . . . .

$$q^n = (E - E_0^*)\frac{\sum_{s=1}^{k} \alpha_s^n \frac{a_{1,s}}{a_{s,1}}\mu_s}{k} + q_0^n,$$

where $k = p^1\sum_{s=1}^{k}\alpha_s^1(a_{1,s}/a_{s,1})\mu_s + \cdots + p^n\sum_{s=1}^{k}\alpha_s^n(a_{1,s}/a_{s,1})\mu_s$ and E denotes the income corresponding to $q = (q^1, \cdots, q^n)$. Hence the slopes of the Engel curve C are given by the expressions:

$$\alpha^1 = \frac{\sum_{s=1}^{k} \alpha_s^1 \frac{a_{1,s}}{a_{s,1}}\mu_s}{k},$$

. . . . . .

$$\alpha^n = \frac{\sum_{s=1}^{k} \alpha_s^n \frac{a_{1,s}}{a_{s,1}}\mu_s}{k}.$$

Since each straight line parallel to C which goes through a point of S^* is also an Engel curve corresponding to the price vector p, we can summarize our results in the following:

THEOREM X. *Let us consider k time periods, $t_1, \cdots, t_k$, and the corresponding Engel curves $C_1, \cdots, C_k$. Let the indicator be a polynomial of the second degree, and let the Engel curve C_i be given by the equations*

$$q^1 = \alpha_i^1 E_i + \beta_i^1, \qquad q^2 = \alpha_i^2 E_i + \beta_i^2, \qquad \cdots, \qquad q^n = \alpha_i^n E_i + \beta_i^n.$$

Denote the price vector in the period t_i by $p_i = (p_i^1, \cdots, p_i^n)$. If $p = (p^1, \cdots, p^n)$ is a price vector equal to the vector sum $\mu_1 p_1 + \cdots + \mu_k p_k$, then the straight line with the slopes,

$$(59) \qquad \alpha^1 = \frac{\sum_{s=1}^{k} \alpha_s^1 \frac{a_{1,s}}{a_{s,1}}\mu_s}{k}, \qquad \cdots, \qquad \alpha^n = \frac{\sum_{s=1}^{k} \alpha_s^n \frac{a_{1,s}}{a_{s,1}}\mu_s}{k},$$

and going through a point of S^* (S^* is the space where the marginal utility of money is equal to zero) is an Engel curve corresponding to the price vector p. The quantities $\alpha_s^1, \cdots, \alpha_s^n$ denote the slopes of the Engel curves C_s,

$$a_{i,j} = \sum_{r=1}^{n} \alpha_i^r p_j^r \text{ and } k = p^1\sum \alpha_s^1 \frac{a_{1,s}}{a_{s,1}}\mu_s + \cdots + p^n\sum \alpha_s^n \frac{a_{1,s}}{a_{s,1}}\mu_s.$$

If the Engel curves $C_1, \cdots, C_k$ have a common intersection point q_0, then S^* reduces to q_0 and the Engel curve is uniquely determined. Then we can say: The straight line with the slopes given by (59) *and going through q_0 is the Engel curve corresponding to the price vector p.*

2. *The calculation of the index of cost of living.* The determination of the indifference surfaces enables us to calculate the index of cost of living. The problem of the index is as follows: we must calculate for each given income E_1 in the first period the equivalent income E_2 in the second period. At first we determine the point q_1 of the Engel curve C_1 corresponding to the income E_1. Since the indicator $I(q)$ can be calculated in the linear space determined by the Engel curves C_1 and C_2, we can determine the point q_2 of the Engel curve C_2 for which $I(q_2) = I(q_1)$. The value E_2 of q_2 calculated with the prices of the second period gives us the equivalent income to E_1 and the index is given by E_2/E_1. If we make these calculations we get the same formula for the index, already demonstrated in another way.[7]

3. *The case when the integrability conditions are not fulfilled.* If the integrability conditions are not fulfilled, at least one of our assumptions must be given up. There arises the question, which of our assumptions is to be rejected. Especially one may ask, whether the assumption that the indicator can be approximated by a polynomial of the second degree is to be rejected, or rather the assumption that there exists a system of indifference surfaces which remained unchanged during the period of observations? If we consider only a region T of the quantity space, where the Engel curves can be well approximated by straight lines, and if the diameter of T is not much greater than the greatest length of the parts of the Engel curves contained in T, then we have all reason to assume that the indicator $I(q)$, provided that one exists, can be approximated by a polynomial of the second degree. Hence in such cases we are justified in saying that the hypothesis of the existence of a system of indifference surfaces which remained unchanged during the period of observations is to be rejected. Hence this method enables us to examine whether the indifference surfaces of a certain group of individuals remained unchanged during the period of observations, or whether different groups of individuals observed in the same time period have an identical system of indifference surfaces.

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[7] See the writer's paper on index numbers in *Zeitschrift für Nationalökonomie*, Wien, 1937.

REPORT OF THE PHILADELPHIA MEETING, DECEMBER 27–29, 1939

THE AMERICAN WINTER MEETING of the Econometric Society was held at Philadelphia, Pennsylvania, December 27–29, 1939, in connection with the meetings of the other social-science societies. The sessions were held in the Ritz-Carlton, the Walton, and the Bellevue-Stratford Hotels, the last of which was the headquarters of the Society. There was excellent attendance in spite of the competition of numerous other sessions held at the same time.

The opening session, on Wednesday morning, December 27, was devoted to National Income and Employment. The chairman was Professor Elmer C. Bratt of Lehigh University, and the following papers were presented:

LOUIS H. BEAN, Agricultural Adjustment Administration, *National Income, Production, and Unemployment: Uses and Prospects.*—Mr. Bean dealt with the quantitative relationship between industrial production, employment, expenditures for new durable goods, and the national income. One projection of industrial production for 1940 points to a level of activity for that year of 8 to 12 per cent above the average for 1939. This would mean a smaller letdown from the present peak volume than a number of analysts expect. It would mean a total of nonagricultural employment of 36 to 36.5 millions compared with 35 at present. It would also mean an increase in expenditures for new durable goods by producers and consumers from 16 billions in 1938 to perhaps 21 billions in 1940. This would be an increase for 1940 over 1939 about as large as the increase of 1939 over that of 1938. Furthermore, for every billion dollars of expenditures for new durable goods the national income tends to increase by about two and a quarter billion dollars. On this basis the national income for 1940 might be close to 74 billion dollars compared with something less than 70 billions in 1939, 66 billions in 1938, and over 80 billions in 1929. Among new economic data involved in these analyses two new monthly series were presented, one on total nonagricultural employment, another on nonagricultural income from 1919 to 1929 in continuation of the similar series now published currently by the Bureau of Labor Statistics and the Department of Commerce.

DAVID WEINTRAUB and HARRY MAGDOFF, W.P.A. National Research Project, *The Service Industries in Relation to Employment Trends.*—Messrs. Weintraub and Magdoff pointed out that in 1930 over 40 per cent of those gainfully employed were attached to service industries, and that this proportion had been growing steadily for a long time and rapidly during the nineteen-twenties. Of those employed

in service industries in 1930, about 52 per cent were attached to industries concerned with the production, distribution, or financing of commodities; and thus the trend of employment in this field depends upon the trend of commodity production. Over one-eighth were attached to the public services; although these services are ultimately dependent on the incomes derived from the production of other goods and services, the need for public services and the pressure to borrow or tax may at times change in reverse order to the functioning of private business. About one-third were engaged in domestic and personal services, which depend primarily on the employers' income from other activities and on the degree to which various household functions can be supplied less expensively on a commercial basis. Historically, the growth of employment in services has depended on the increasing ability of the population to produce the necessaries of life with a decreasing proportion of the working population, thus releasing an increasing proportion for production of other commodities and services which permit a rising standard of living. This type of social progress has limits set by technological factors and also by economic relationships. The continued flow of the increased purchasing power required to support a growing volume of goods and services depends on increasing investments and the periodic revitalization of investments in the production of producers' durable goods. Whenever such investments slow down, the production of goods slows down and with them the related services. It therefore seems futile to look for continuous expansion of employment in the service industries without first positing a continuous expansion of investments in capital goods and an expansion of the production of those goods and services in which these capital goods are to be used.

The afternoon session on Wednesday was held jointly with the American Statistical Association and the Advisory Committee on Research in Finance. The subject was the Role of Consumer Financing in Economic Organization. The chair was taken by Dr. W. F. Kurtz of the Russell Sage Foundation and the papers were as follows:

ROLF NUGENT, Russell Sage Foundation, *Pattern of Consumer Instalment Debt.*—Dr. Nugent presented estimates of open-book and instalment credit outstandings, subject to various qualifications, which cannot be condensed into a brief abstract.

DAVID DURAND, National Bureau of Economic Research, *Risk Elements in Consumer Instalment Financing.*—Dr. Durand discussed in general the feasibility of applying statistical methods to the problems of credit analysis and in particular the use of statistical sampling techniques. Probably the most efficient method is to obtain two separate samples of approximately equal size, one representing the "good,"

or more satisfactory transactions, the other representing the "bad," or less satisfactory transactions. Although this approach is subject to definite limitations, it nevertheless gives results. Its application by the National Bureau of Economic Research to samples obtained from a variety of consumer-financing agencies indicates that some borrower characteristics are definitely and rather universally related to credit experience. Stability of employment, for example, indicated better than average risk in 12 commercial-bank samples, 3 industrial-bank samples, and 4 automobile-finance samples—which comprised all the information available. An attempt was made—using one of the most recently developed techniques—to combine stability of employment and other significant factors into a combined index of risk. The attempt was successful—at least in the sense that the index was a better credit indicator than any of the component parts. Although several of the factors investigated are undoubtedly related to risk, it is very doubtful that any of them singly, or a combination of them, is sufficiently closely related to provide a reliable means of prediction. A rough index of efficiency suggested that even the best of credit indicators is a long way from perfect. We may infer from this either that credit managers are doing so well as to leave no room for improvement; or that the statistical techniques so far employed by the Bureau are inadequate for the problem; or that the problems of credit management cannot be reduced to an exact science.

GOTTFRIED HABERLER, Harvard University, *Consumer Financing and Economic Fluctuations.*

On Thursday morning, December 28, the topic of Monetary Processes in a War Economy was discussed, under the chairmanship of Professor James W. Angell of Columbia University. The papers were:

JOSEPH A. SCHUMPETER, Harvard University, *Monetary Processes during War Booms and Postwar Slumps.*—Professor Schumpeter said that a war boom occurs if the impact of war expenditure more than compensates for the effects of the disturbances and restrictions incident to war, a postwar slump if the effects of expected reductions in public expenditure are not more than compensated by the expenditure incident to the normalizing processes. It follows that war booms and postwar slumps cannot be expected with certainty and that there might be war slumps and postwar booms instead. While war booms and postwar slumps bear, especially as to their monetary aspects, a strong similarity to ordinary business booms and slumps, particularly to those of the Juglar cycle, this similarity is only on the surface, whereas the difference is fundamental. The investment activity of the ordinary business boom produces the results which then account for the phenomena of recession and depression. The war boom being funda-

mentally nothing but an excess in consumption and leaving the industrial apparatus in an antiquated and drowsy state, produces only impoverishment and a sickly growth of war industries. A subsequent slump is, therefore, not analogous to an ordinary business recession except in the general feature that it is also a phenomenon of adaptation after disturbance. The inference is that there is a much stronger argument for suppressing war booms than there is for suppressing ordinary booms (for instance, in the way suggested by Mr. Keynes, the principal thing is to keep the wage bill down and to tax heavily not undistributed but precisely distributed profits) and that the way to manage war slumps is to reduce taxation on enterprise, to give a lead in those directions in which expansion after a prolonged war can most confidently be expected (housing and so on) and to leave monetary processes and foreign exchanges for the time being to themselves.

HORST MENDERSHAUSEN, Colorado College, *Changing Methods of Diverting the National Product to War Purposes.*—Dr. Mendershausen pointed out that a country's economy is operated in times of peace on the basis of two sets of guiding principles. Consumers, with unequal incomes, try to derive maximum satisfaction from the disposition of their incomes. Entrepreneurs and capitalists try to realize maximum profits, to maintain and to increase the value of their individual capitals. If the country becomes involved in a modern major war, these guiding principles are likely to conflict with the new social aim of winning the war. Left to itself, the economy will effectuate the necessary diversion of a large part of the national product to war purposes by processes which are likely to be slow and chaotic and which imply a high degree of inequality in the distribution of scarce but essential foods. In order to make citizens share the burden of war according to the *maximum* sacrifice each can make, and in order to ensure a rational distribution of the burden over the expenditures or consumption on the one hand, and over capital maintenance and increase on the other hand, government intervention is unavoidable. Two types of policy may be distinguished, the therapeutic and the prophylactic. Some kind of therapeutic policy has been followed by all countries during the first World War. Relying on the economy's ability to adapt itself to war conditions, the governments permitted a considerable rise of the level of prices and of certain incomes, supporting these movements by inflationary measures. The way scarce essentials were distributed endangered public health and morale. Corrective measures were late, incomplete and badly co-ordinated. Prophylactic policies are attempted by several countries in the present war. Early and comprehensive fixing of maximum prices is one of their characteristics. One economic problem consists in distinguishing between such price in-

creases as are necessary to cover increasing costs in terms of effort and others that are due to profiteering. An organized reduction of depreciation allowances may help to reduce the rise of certain prices. Principles of price fixing such as the "bulkline" principle have to be developed. Early reinforcement of a strongly progressive income taxation together with excise taxes on unnecessary commodities are further features of a prophylactic policy. To prevent the subsisting inequality of incomes from affecting the supply of scarce essentials to the poorer classes, rationing seems the best means. A prophylactic policy makes possible a more transparent and rational distribution of war burdens. Its advantages are the more obvious the more expensive the war in proportion to the national income and wealth.

The ensuing discussion was opened by Dr. PAUL M. SWEEZY of Harvard University, who expressed general agreement with Dr. Mendershausen's paper but examined Professor Schumpeter's propositions more critically as follows: He thought that war often induced wide technical advances in industry, instead of technical lethargy; that the lag of reported price changes behind changes in money stock was largely due to wartime price controls; that the effect of priorities and the like in rationing goods (instead of depending on price rises to accomplish such selections) had been overlooked; that war often improved the aggregate utilization of resources by eliminating such wastes as competitive advertising; that the effect of the wartime destruction of real capital and resources, which often accounts for the postwar monetary degeneration, had been ignored; and that the contention that large wartime profits should be allowed provided the recipients were *not* allowed to spend them was dubious, since such profits could be expended on goods in competition with the government even if they were not actually disbursed to individuals. Professor SEYMOUR E. HARRIS of Harvard University also stressed the fact that wartime price controls may deprive the index of reported prices, and hence its lags of any great significance. Mr. A. P. LERNER of the London School of Economics favored high taxes or increased borrowing in wartime, to remove from the market as much as possible of the effective demand which would otherwise compete with the government. Mr. REDVERS OPIE of the University of Oxford likewise favored increased taxes; and had the same misgivings as the other discussants concerning the significance of any general price index, as a guide to policy, in an era of price controls.

On Thursday afternoon a joint session was held with the Institute of Mathematical Statistics on Recent Advances in Business-Cycle Analysis. Professor Harold T. Davis of Northwestern University was in the chair and the following papers were given:

MERRILL M. FLOOD, Princeton University, *Recursive Methods in Business-Cycle Analysis.*—Dr. Flood pointed out that the prediction of tax-revenue yields for purposes of budget control requires quantitative knowledge of the business cycle. J. Tinbergen,[1] and others, have constructed complete mathematical business-cycle theories such that some, or all, of the economic series considered satisfy linear difference equations. This observation is taken as cause sufficient to justify an empirical examination of certain economic time series to determine linear stochastical difference equations which they satisfy, and the extent to which such equations may be used for forecasting purposes. H. Wold[2] has given a detailed discussion of the mathematical aspects of this problem and has illustrated the methods by applications to economic series, using the processes of hidden periodicities, linear autoregression, and moving averages. The linear difference equations obtained by applications of these processes, when used recursively, provide forecasts which may be compared with others obtained directly by regression analysis, to determine the consistency of the system of forecasting equations. These comparisons, when made for one particular series studied, indicated, for example, that a fifth-order difference equation was superior to one of lower order even though both appeared equally efficient when estimating in the same period as that used in their derivation. In many other instances significant differences were found between the standard deviations of estimate in the data period used in the derivation of the difference equation and a later testing period. Time periods of varying lengths, though only slightly different, often gave equations quite dissimilar but some similarity was observed in the cycles obtained. The methods used, when extended to the case of several series considered simultaneously, provide a very general framework for empirical analysis. Further progress based on such empirical analysis awaits the development of simpler computational methods and a more adequate theoretical structure to indicate which economic series are related in this mathematical fashion.

GERHARD TINTNER, Iowa State College, *An Appreciation of Some Recent Mathematical Business-Cycle Theories.*—Professor Tintner described briefly three recent mathematical business-cycle theories, presented by Frisch, Tinbergen, and Kalecki. All three lead to difference-differential equations which may have oscillatory solutions. They are all criticized because: (1) "macrodynamic" economic systems imply extensive simplifications, whose economic meaning and justification

[1] J. Tinbergen, ECONOMETRICA, Vol. 3, July, 1935, pp. 241–308; Vol. 6, January, 1938, pp. 22–39.

[2] H. Wold, *A Study in the Analysis of Stationary Time Series*, Uppsala, 1938.

is not always apparent, (2) the role of expectations and anticipations is not made explicit, (3) cycle and trend are probably so closely related that they should be studied together. The statistical verification attempted by Tinbergen is criticized because of the difficulties involved in correlating time series. It is doubtful whether the findings are more than a very rough description of historical data. Great caution should be exercised in using them as a basis for cycle policy.

In the discussion Mr. ABRAM BERGSON (BURK) of Harvard University said that, while his general reaction to the theories Professor Tintner discussed was sympathetic to his, he was gloomy as to the alternative approach suggested. The experience of Mr. Hicks, whose work Tintner regards as a likely model, is hardly encouraging. Hicks' endeavor to extend the Walrasian analysis to dynamic economics results in few empiric propositions of interest, and the assumption of perfect competition, which the author states is necessary to the analysis, hardly inspires confidence. Whether or not the difficulties confronting Hicks can be avoided, it does seem in any case that Mr. Tintner imposes an unnecessary burden. Everyone agrees that it is desirable to state assumptions in such a form that their appropriateness can be judged, and that the assumptions made should have as much support as possible. But, in view of the fact that one of the variables which economists must economize is time, the selection of a few strategic variables for analysis at the present stage of business-cycle theory seems clearly warranted. Working now with (say) n consumers' goods rather than one, or n producers' goods rather than one, it seems clear, will not yield additional fruits worth the effort in business-cycle analysis. Tintner's objections to the macrodynamic theories seem valid in so far as he refers to the type of assumptions introduced, but not in so far as he objects to the fact that they are *macro*dynamic.

ARNE FISHER, New York City, *The Statistician's New Clothes, Being a Trivial Discourse by a Layman on Certain Fables and Fairy Tales Found in the Discussions of the So-called Business Cycles in Recently Published College Texts on Statistical Methods.*—Mr. Fisher pointed out that it is difficult to summarize the extensive material on business cycles and that even the definition is uncertain. There are many complexities. Professor Schumpeter, for example, says:

> The trend we decide on will determine what the fluctuations will be. But irregularities and cyclical fluctuations will, in turn, determine the trend. It follows that no progress towards analysis is possible on these lines.[3]

Other writers are less intelligible. The supply of hypotheses and quaint technical labels is beginning to run out. Colonel Leonard P. Ayres

[3] J. A. Schumpeter, *Business Cycles*, New York, 1939, p. 202.

claims to have determined the so-called "trend" of American business enterprise, almost from the time of the Revolution, by the method of least squares. A careful reading of the Colonel's text fails, however, to disclose that this is based on critical knowledge of the philosophical and mathematical principles of the method itself. In a discussion of the theory of business cycles one should not overlook Professor F. C. Mills' lucid text on *Statistical Methods*. Briefly, and in a somewhat condensed form, his treatment of time series and business cycles may be stated as follows: Economic phenomena when arranged chronologically or as time series can be represented in mathematical symbolism in the form

$$Q_t[L(t) + C(t)] + \Delta = o(t), \text{ where}$$

Q_t is a sort of periodically recurring seasonal factor,
$L(t)$ is the secular or long-term trend,
$C(t)$ is the cyclical movement.

Both $L(t)$ and $C(t)$ are supposed to be some sort of functions of the time element, t, while the symbol Δ denotes chance or accidental fluctuations (i.e., residuals) which are independent of time. The symbol $o(t)$ is the observed phenomenon at time t. Provided we can express the symbols $L(t)$ and $C(t)$ as some plausible mathematical functions of t, the least-squares method criterion is that $\sum\{Q_t[L(t)+C(t)]-o(t)\}^2 = \sum\Delta^2$ must equal a minimum. Now what the academic headmasters actually do is: 1. To determine some crude empirical correction factor for Q_t; 2. To drop quite casually, or *sub rosa* so to speak, the term $C(t)$; and 3. To write $Q_tL(t)$ as $a+bt$ or some other simple parabolic or exponential expression. Thus the secular trend takes on the least-squares form of $\sum[a+bt-o(t)]^2=\sum\Delta^2$ must equal a minimum. To be sure, no matter how charmingly improbable such procedure is, it will invariably furnish some numerical values of the constants a and b, which will satisfy the above condition that the sum of the squares of the residuals becomes a minimum. But, as pointed out by Thiele, whether this sum is the correct one or some other, even an incorrect sum, is completely ignored. The next step is to take the percentage residuals from this "trend" as constituting the business cycle. Professor Mills admits the weakness of this method, saying that in "deriving, finally, a set of residual figures which are taken to reflect the cyclical fluctuations in an economic series we are, of course, abstracting from reality." This is but a polite way of saying that the whole pretty theory of the analysis of business cycles is void of sense, an empty abstraction. It suggests a brief definition of the so-called business cycle as follows: A business cycle is a chronologically arranged sequence of chance errors; or, in more explicit form: The error residuals, resulting from an

improbable theory, when arranged in chronological order constitute a business cycle.

On Thursday evening Statistical Investigations Pertaining to Cycle Theory were discussed, under the chairmanship of Professor Joseph A. Schumpeter of Harvard University, in the following papers:

J. B. D. DERKSEN, Netherlands Statistical Institute, *Long Cycles in Residential Building: An Explanation.*—This paper is printed in full in this issue of ECONOMETRICA.

HANS STAEHLE, Harvard University, *Elasticity of Demand and Social Welfare.*[4]—Dr. Staehle remarked that the comparative effect of changes in the prices of different individual commodities upon welfare is usually assessed by means of the consumers'-surplus apparatus. This tool would lead to the conclusion that the gain in welfare due to a price reduction will be the greater, and the loss in welfare due to a price increase the smaller, the larger the (price) elasticity of demand for the commodity in question. This, however, applies only to single individuals and to changes of such scale as will leave the marginal utility of money unaltered. When applied to market demand curves and to such changes in price as are large enough, and possibly designed, to influence people's welfare (i.e., to modify the marginal utility of money to them), the consumers'-surplus conclusion loses its validity. An alternative approach is therefore suggested, on the basis of the following propositions: (1) Though there exists no necessary general relationship between price and income elasticity of demand, it is probable that their coefficients are of like value whenever income elasticity is found to be zero or very small over a fairly wide range of incomes, including very low levels. It can indeed be shown by family-budget statistics that the coefficient of income elasticity of demand for any given commodity has a tendency to decrease while income increases. Consequently, if a very small coefficient is found even on the lowest income levels, the commodity probably both satisfies an urgent need and has no substitutes. (2) In the case of a highly homogeneous, though small, group of families whose income per "unit" was very nearly the same and of very low level, a positive correlation was found to exist between (a) the income elasticity of demand (measured from the average consumption among the group) for each of 26 commodities, and (b) the coefficient of variation of the quantities of each commodity consumed per "unit" by the various families. This suggests that income elasticity coefficients are the more reliable as indicators of consumers' behavior in the event of an income change, the lower their value. (3) For any given frequency distribution of incomes the addition of an

[4] The full paper will be published in the *Quarterly Journal of Economics.*

equal absolute money sum to every income diminishes, and the subtraction of an equal absolute money sum increases, the degree of inequality of the distribution. If it is agreed, moreover, that any economic change is to be considered socially desirable if it decreases, and undesirable if it increases, the inequality in the distribution of incomes, it follows that price reductions are the more (and price increases the less) desirable from a social viewpoint, the lower the income elasticity (and thus the price elasticity) of demand for the commodities in question *on the lowest income levels.* The advantage of a price reduction for such commodities accrues more evenly to consumers both on a given and on different income levels and thus is equivalent to the addition of a fixed money amount to every income which diminishes the inequality of the income distribution. Conversely, the burden of a price rise for such a commodity would translate itself by an increase in the inequality of the distribution of (real) incomes and thus be socially more undesirable than the rise in the price of some other commodity.

The Friday morning session, December 29, was on the Theory of Employment, with Professor Alvin H. Hansen of Harvard University in the chair. The following papers were presented:

Richard M. Bissell, Jr., Yale University, *Wage and Price Policies and the Theory of Employment.*—Professor Bissell's paper was concerned with the controversial subject of the effect upon the volume of employment in the community of changes in wage rates. The method employed was that of income analysis, that is, the study of the effects upon receipts and disbursements by various groups in the community of changes in prices, wage rates, etc., a method somewhat similar to that employed by Keynes. The results of wage-rate reductions were treated under various assumptions as to the behavior of prices. In the discussion of the first two cases treated, it was assumed that disbursements of business firms in the form of dividend or interest payments and withdrawals from businesses to cover interest and profits would remain in the short run unaffected by changes in business profits. Employing this assumption, it was demonstrated that a general reduction in wages, assuming prices to be flexible, would give rise to an increase in output and employment because it would constitute an increase in the real income disbursed in the form of interest and dividends. It would also involve, however, a decline in profits or an increase in losses and, therefore, dissaving by business firms. If it is assumed that prices are entirely inflexible, different results will follow. In this case, output and employment will shrink and business saving increase. These conclusions differ from those reached by Keynes only because of the assumption that interest and dividend disbursements do not

vary in the short run. Thus, the results of the wage changes cannot be expected to be stable because in the longer run interest and dividend payments will reflect the level of earnings. The later sections of the paper were devoted to a discussion of the effects of relative wage and price changes and to the way in which they can be dealt with in the framework of income analysis.

ARTHUR SMITHIES, University of Michigan, *Dynamic Adjustments in Imperfect Competition.*—Professor Smithies determined the rates of production over a period of time by which two (identical) competitors in an imperfect market, who are assumed not to be in a position of equilibrium at the outset, adapt themselves to a given demand situation. It is assumed that each competitor must plan production 1 unit of time in advance, and that, in order to plan production at a given point of time, he makes an estimate of the price his rival will be charging at that point. This estimated price is assumed to be a function of the present prices of the two producers. It is then possible to express the rate of expected profits at time t as follows:

$$\pi_1 = f_1[x(t), x(t-1), y(t-)],$$
$$\pi_2 = f_2[y(t), y(t-1), x(t-1)],$$

where x and y denote rates of production of the two competitors. Total expected profits over a period, which, it is assumed, the competitors attempt to maximize simultaneously, are then the integrals of these functions over the period. The solution of the problem involves applying the technique of the calculus of variations to these integrals, and it is found that, for the special linear-demand, marginal-cost, and expectation functions assumed, the optimum shapes of x and y are the solutions of two linear second-order difference equations, specialized so as to fit arbitrary initial conditions and to satisfy certain terminal conditions relating to the end of the period. x and y are found to be roughly cyclical functions of t, with a period about twice the planning period. If the period of adaptation is allowed to approach infinity, it is found that in general the cycles become damped, and a stable equilibrium is eventually achieved. The degree of damping depends inversely on the extent to which each competitor expects his rival to meet his price. Although the paper is restricted to a special case for the sake of achieving qualitative results, the methods used are general.

MONTGOMERY D. ANDERSON, University of Florida, *Dynamics in the Theory of Employment.*—Professor Anderson developed the following formula for total employment in a closed economy:

$$N = \frac{MV}{(\mu + 1)PX'}$$

where N is the equivalent number of persons employed "full" time, M is the amount of money in circulation (including bank deposits), V is the average rate of turnover of money in exchange for reproducible goods and labor, P is an index of the price level, X is the average "real" wage per person per interval of time, and μ is the (weighted) average number of times a bundle of labor services changes hands between the point at which the services are applied to capital and the point at which the capital and labor emerge as "finished" goods. The application of this formula to data for the United States yields results very similar to those obtained by Tinbergen and De Wolff for consumer-goods industries in the same country.[5] The long-run value of the coefficient μ in the formula just given measures the complexity of industrial processes; in the short run μ furnishes a rough measure of speculation in goods. By making certain plausible hypotheses it was shown that the formula for employment can be reduced to a differential equation with only two independent variables. The accuracy of this reduced equation is attested by a multiple-correlation coefficient of $r=0.95$ with respect to a recent ten-year period.

On Friday afternoon a joint session was held with the American Economic Association on the subject of Cost Functions and Their Relation to Imperfect Competition. The chairman was Professor Theodore O. Yntema of the University of Chicago and the Cowles Commission for Research in Economics, and the papers were as follows:

J. M. CLARK, Columbia University, *Some Speculations on the Requirements of Workable as Distinct from Ideal Competition.*—Professor Clark stated that where one of the conditions of perfect competition is absent (as is always the case) the presence of others may lead to greater rather than less imperfection. Potential competition remains an important factor. So far as imperfect competition is governed by long-run curves of individual demand and cost, these are flatter than commonly represented, and the imperfection of competition correspondingly less. As to short-run problems, industry with fluctuating demand requires prices in excess of short-run marginal cost, though still limited by competition. Favorable conditions appear to include a sloping individual-demand curve, and some uncertainty whether a reduction of price will be quickly met. With standardized products, a chaotic market tends toward cutthroat competition. Oligopoly with a standard product is limited by its one-way action, by the chance to gain good will or large orders through initiating a price reduction, by

[5] See J. Tinbergen and P. De Wolff, "A Simplified Model of the Causation of Technological Unemployment," ECONOMETRICA, Vol. 7, July, 1939, pp. 199 ff.

the personal equations of producers, and by some intermittent "chiselling." A special case, not wholly impossible, is that of a standard product, sloping individual-demand curve, and some uncertainty as to the meeting of price reductions. While extreme cases of quality differentials approach monopoly, more moderate cases afford the conditions of fairly healthy and workable imperfect competition. This class of cases may be increased in the future by the further development of closer substitutes for existing products, together with better knowledge of qualities and specifications on the part of buyers.

JOEL DEAN, the University of Chicago and Cowles Commission for Research in Economics, *Statistical Cost Curves in Various Industries.*—Professor Dean presented provisional findings of an investigation designed to determine empirically the patterns of short-run and long-run cost behavior of individual enterprises in a variety of industries. Six case studies of short-run cost were summarized. To observe the effects upon cost of variation in rate of plant utilization, firms and observation periods were selected that displayed great variability of output while eliminating or minimizing changes in scale, technology, proportion and quality of products, and price-motivated input substitution. When necessary, an index of output was constructed. Heterogeneous products were combined by weighting each according to its deflated standard cost. Recorded components of combined cost were individually rectified for changes in prices of materials and services, for errors in accounting allocation, and for discrepancies between the time of recording output and the costs that contributed to it. Multiple-correlation analysis was used to determine the relationship between these corrected cost elements (individually and in combination) and output, while holding constant at their means other significant cost influences peculiar to each enterprise. The character of the resulting cost functions was validated by critical-ratio tests, by variance analysis, and by independent analysis of cost and output first differences. Typical multiple-correlation results, for a leather-belt shop, three department-store departments, a hosiery mill, and a furniture factory, were shown by a series of slides. Partial regressions of cost on output and on other operating factors were displayed, together with total, average, and marginal statistical cost curves for combined cost and its components. Theoretical analysis usually posits rising marginal cost over the relevant output range, yet no instance of rising marginal cost within the range of actual operation was found. It appears likely, however, that marginal cost will rise beyond this observed range as the limit of physical capacity is reached. Possible explanations are multiple-battery plants, plants designed for minimum cost near physi-

cal capacity, and technological, conventional, or market restraints on cost "forcing" of output.

ROSWELL H. WHITMAN, R. H. Macy & Company, *Cost Functions in the Department Store.*—Dr. Whitman pointed out that cost behavior for department stores is complex. Its study is hampered by problems of determining a measure of "output." The data used for the present paper were weekly costs and output. Output was corrected for non-homogeneity both by a multiple-regression equation and by creating an index with average cost weights. The cost functions presented included: 1. The relation of delivery mileage cost to packages delivered. The marginal cost is a decreasing function of output. 2. The relation of salesclerk costs to the number of transactions. This may be a decreasing, constant, or increasing function of output, depending on the organization of the selling department, the character of its peak business, and so forth. 3. The total-store variable-cost function. This function has constant marginal costs within the normal range of output, but high marginal costs at the peak. This cost behavior is due to a relatively low efficiency as near capacity output is approached. While marginal costs are an increasing function of output, because of the seasonal fluctuation of business and the relatively small seasonal fluctuation in price, price is above marginal costs for most of the year. Though this indicates imperfect competition, it is not necessarily socially undesirable to have eleven months of idle capacity in order to serve the public during the Christmas shopping season.

In the discussion, Professor GEORGE J. STIGLER of the University of Minnesota remarked that the statistical cost curves presented by Messrs. Dean and Whitman seem somewhat in conflict with our expectations, based as they are in large part on the classic law of diminishing returns. In the cases of constant short-run marginal costs, undoubtedly the existence of a divisible plant (many similar machines, for instance) is an important explanation, as Dean suggests. In addition, there are probably two biases in the statistical procedure which lead to the same result in the cases here presented: First, plants which had experienced very wide fluctuations of output were selected, and these are precisely the plants which would be built with a maximum of flexibility (so that the marginal-cost curve would be relatively flat). Second, short-run alterations of plant, which usually cannot be eliminated on the basis of accounting records, lead to a speciously flat marginal-cost curve. The falling marginal-cost curves are even more suspect; they seem to be due to variations of product quality and to misallocation of costs as between fixed and variable. An alternative method of deriving cost curves deserves serious consideration: the

construction of the production function. This method has its defects, not the least of which is that the economist must become a technician. But there are important advantages to this alternative: we are released from dependence on accounting records, which are frequently unavailable and even more often unusable; and the statistical problems involved in time series are largely avoided. Professor Stigler raised only two questions with respect to Professor Clark's very stimulating paper. First, he questioned the general importance of potential competition as a limitation on monopoly prices, arguing that (1) low prices *per se* will not discourage potential rivals, who may expect a quota, and (2) if such a quota is refused, and the existing firms will not tolerate new rivals, there are cheaper methods of communicating this intention to potential rivals than by setting low prices. Second, he could not see why flat marginal-cost curves (well below average costs throughout a considerable range of output) make pricing according to the competitive norm, marginal cost equals price, impossible. For there is demonstrably always such an amount of investment in an industry that, given the fluctuations in demand, the losses and profits will cancel over time. Of course investment is frequently such as to make losses exceed profits if a competitive pricing policy is followed, but it may be argued that this is due to price agreements, and that one cannot justify such agreements on grounds of their necessity to prevent losses because this is circular reasoning.

There ensued an active general discussion in which the speakers and many others among those present participated.

The final session, on Accounting Procedures, was held Friday evening, with the following papers:

JOHN B. CANNING, Stanford University, *Nonadditive Characteristics of Certain Accounting Valuations* (presented by title in the absence of the speaker).—Professor Canning discussed a part of the problem of transforming into realization the hypothesis that a manager strives to maximize the present worth of the concern's future income. To employ productive agents in a manner consistent with such a result requires piecemeal valuations of these facilities. The role of these valuations in guiding managerial policy fixes their characteristics. These include erratic nonadditive properties—in the sense that a sum of such valuations (even within the enterprise context) does not have a meaning, unit for unit, like that of any of the individual valuations. Though by judicious choice of size of the unit asset this erratic effect can be held within bounds, it cannot be eliminated. Balance sheets that reflect the best empirical practices in production programs and in asset appraisal and "going concern" valuation closely approximate

the results implied in theoretical solutions and are subject to the same erratic characteristics. In particular a balance sheet showing a corporate surplus shortly before a bankruptcy which discloses a heavy loss—even to creditors—does not necessarily imply defective piecemeal valuation of assets. Our national indexes of wealth, income, capital formation, etc., are out-of-context summations of such data. To employ such indexes without regard for their erratic character is untidy science.

MARK S. MASSEL,[6] Temporary National Economic Committee, *Discrepancies between Accounting Reports to Governmental Agencies.*—Mr. Massel pointed out that the accounting reports submitted for regulatory or statistical purposes by business organizations to the respective Federal agencies contain marked differences in their essential meaning. For example, a report submitted to the Securities and Exchange Commission by a particular corporation is apt to differ from its report to the Bureau of Internal Revenue, even though the data purport to cover identical operations over an identical period of time. A number of forces lie behind such discrepancies. First, if a business organization is given freedom of choice as to methods of rendering the accounts, self-interest dictates that it vary its approach to the respective agencies. Also, such variations result from differences in the accounting rules of the agencies, which in turn are governed by an agency's primary purposes: to illustrate, one may be interested in how much a certain utility company actually earned during the past year; another, for rate-making purposes, in how much it should earn; and a third, desirous of protecting the investors, in how much it might earn in the future. In some cases, Congress itself determines the rules, but with differences between agencies, and may limit the discretion of one but not of another. Again, there are discrepancies due to court interpretations, or arising simply from the individual opinions of staff members in the agencies. Finally, it should be observed that such forces create conflicting accounting reports to state as compared to Federal agencies or as between state agencies. These differences are significant because they affect valuations of assets, liabilities, and income. From a statistical point of view, summations of the resulting data should be based upon careful analyses of the consistencies and inconsistencies behind the valuations. Otherwise conclusions may be drawn that are at direct variance with the facts. From a regulatory standpoint, the problem is important because the accounting regulations of a given

[6] Mr. Massel is a member of the staff of the Temporary National Economic Committee. The opinions expressed herein are entirely his own, and in no way does this statement present the views of the Committee.

agency may be improved by further examination of those in other agencies. A co-ordinated approach may also be useful to Congress and to the courts. Where possible, differences in the rules might be resolved to the mutual advantage of government and business. This, it must be noted, does not imply that the rules and valuations should necessarily be the same. There is manifest need for a series of studies comparing the rules and the resulting accounting statements. These should be pointed toward reinforcing and synthesizing the regulations and analyzing the possible statistical significance of summary figures.

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